Mathematische Annalen (2020) 376:765–795
https://doi.org/10.1007/s00208-018-01796-y

Mathematische Annalen

Nonlinear interaction of waves in elastodynamics and an inverse problem

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Received: 11 July 2018 / Revised: 8 December 2018 / Published online: 18 January 2019

Abstract

We consider nonlinear elastic wave equations generalizing Gol'dberg's five constants model. We analyze the nonlinear interaction of two distorted plane waves and characterize the possible nonlinear responses. Using the boundary measurements of the nonlinear responses, we solve the inverse problem of determining elastic parameters from the displacement-to-traction map.

1 Introduction

1.1 The nonlinearity in elastodynamics

We introduce the nonlinear elastic system to be studied in this work. Our model is a generalization of the five constant model widely used in the literature since the work of Gol'dberg [4]. We shall follow the presentation in Landau–Lifschitz [12]. The materials are classical, however we would like to review its derivation to show the sources and significance of the nonlinearity in elastodynamics.

Communicated by Y. Giga.

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Consider an elastic body occupying an open bounded region $\Omega \subset \mathbb{R}^3$ with smooth connected boundary $\partial\Omega$. The closure is denoted by $\overline{\Omega}$. We denote points in $\mathbb{R}^3$ by $x = (x_1, x_2, x_3)$. When the body is deformed, the distances between points are changed. Suppose that point $x \in \Omega$ is displaced to $x' = (x_1', x_2', x_3') \in \mathbb{R}^3$ and the displacement vector is $u = x' - x$. The length element $dl = (dx_1 + dx_2 + dx_3)^{\frac{1}{2}}$ is changed to $dl' = (dx_1' + dx_2' + dx_3')^{\frac{1}{2}}$ and

$$(dl')^2 = dl^2 + 2e_{ik}dx_i dx_k,$$

where e_{ik} is the *strain tensor* defined by

$$e_{mn} = \frac{1}{2}\left(\frac{\partial u_m}{\partial x_n} + \frac{\partial u_n}{\partial x_m} + \frac{\partial u_k}{\partial x_m}\frac{\partial u_k}{\partial x_n}\right). \tag{1}$$

Hereafter, the Einstein summation convention is used. The strain tensor describes the changes in an element of length when the body is under deformation. For small deformations, one ignores the quadratic terms and take

$$\widetilde{e}_{mn} = \frac{1}{2}\left(\frac{\partial u_m}{\partial x_n} + \frac{\partial u_n}{\partial x_m}\right)$$

as an approximation of e_{mn}. This is the strain tensor used in linearized elasticity.

We only consider the thermostatic state of the body so that the free energy $\mathscr{E}$ of the body is a scalar function of the strain tensor only, namely $\mathscr{E} = \mathscr{E}(e_{ik})$. For an isotropic elastic medium, we can express $\mathscr{E}$ in terms of the invariants $\mathrm{Tr}(e)$, $\mathrm{Tr}(e^2)$, $\mathrm{Tr}(e^3)$ etc. For small deformation, one expand $\mathscr{E}$ up to quadratic terms in ∇u to get

$$\mathscr{E} = \mathscr{E}_0 + \frac{1}{2}\lambda(x)(\mathrm{Tr}\,\widetilde{e})^2 + \mu(x)\,\mathrm{Tr}(\widetilde{e}^2) = \mathscr{E}_0 + \frac{1}{2}\lambda(x)(\widetilde{e}_{ii})^2 + \mu(x)\widetilde{e}_{ik}^2,$$

where $\mathscr{E}_0$ is a constant and λ, μ are called Lamé coefficients. Note that the $\widetilde{e}_{ik}$ above are indeed e_{ik} as the higher order terms are ignored. The *stress tensor* is given by

$$\widetilde{S}_{mn} = \frac{\partial \mathscr{E}}{\partial \widetilde{e}_{mn}} = \lambda(x)\widetilde{e}_{ii}\delta_{mn} + 2\mu(x)\widetilde{e}_{mn}. \tag{2}$$

To show the dependence of $\widetilde{S}$ on $x \in \mathbb{R}^3$ and u, we also use the notation $\widetilde{S}(x, u)$. The stress tensor is related to the internal force T of the body under deformation via $T = \nabla \cdot \widetilde{S}$. Now using Newton's second law, we obtain the differential equation describing the deformation of the body

$$\rho\frac{\partial^2 u}{\partial t^2} = \nabla \cdot \widetilde{S}(x, u) + F, \tag{3}$$

where $F = (F_1, F_2, F_3) \in \mathbb{R}^3$ is an (external) force on the body (e.g. the gravity) and ρ is the density of the elastic medium. Actually, we just derived the linearized elastic wave equation.

Now we take into account the nonlinear effects. We expand the energy density $\mathscr{E}$ to cubic terms

$$\begin{aligned}\mathscr{E} &= \mathscr{E}_0 + \frac{1}{2}\lambda(x)(\operatorname{Tr} e)^2 + \mu(x)\operatorname{Tr}(e^2)\\ &\quad + \frac{1}{3}A(x)\operatorname{Tr}(e^3) + B(x)\operatorname{Tr}(e^2)\operatorname{Tr}(e) + \frac{1}{3}C(x)(\operatorname{Tr} e)^3\\ &= \mathscr{E}_0 + \frac{1}{2}\lambda(x)(e_{ii})^2 + \mu(x)e_{ik}^2 + \frac{1}{3}A(x)e_{ik}e_{il}e_{kl} + B(x)e_{ik}^2 e_{ll} + \frac{1}{3}C(x)(e_{ll})^3,\end{aligned}$$

see Landau–Lifschitz [12, Section 26]. In the reference, λ, μ, A, B, C are all constants so the model is called the five constant model. Other equivalent forms in the literature and their relations can be found in Norris [17]. Here, we consider a more general model in which all the parameters are smooth functions on $\overline{\Omega}$. In the expression of $\mathscr{E}$, we should use the strain tensor in (1) and keep the nonlinear terms. We consider the tensor defined as

$$\begin{aligned}S_{mn} = \frac{\partial \mathscr{E}}{\partial(\partial u_m/\partial x_n)} &= \lambda(x)e_{jj}\left(\delta_{mn} + \frac{\partial u_m}{\partial x_n}\right) + 2\mu(x)\left(e_{nm} + e_{nj}\frac{\partial u_m}{\partial x_j}\right)\\ &\quad + A(x)e_{mj}e_{nj} + B(x)(2e_{jj}e_{mn} + e_{ij}e_{ij}\delta_{mn}) + C(x)e_{ii}e_{jj}\delta_{mn}, \quad m, n = 1, 2, 3.\end{aligned} \tag{4}$$

This tensor is no longer the stress tensor and it is not symmetric. However, the quantity $\nabla \cdot S$ still gives the internal force, hence we again get the dynamical equation of the same form

$$\rho\frac{\partial^2 u}{\partial t^2} = \nabla \cdot S(x, u) + F. \tag{5}$$

This is the nonlinear elastic equation we study in this work. We point out that the nonlinearity of the system comes from two sources: the higher order expansion of the free energy $\mathscr{E}$ and the nonlinear term in the strain tensor.

1.2 The interaction of two waves

We consider the initial boundary value problem for (5):

$$\begin{aligned}&\frac{\partial^2 u(t,x)}{\partial t^2} - \nabla \cdot S(x, u(t,x)) = 0, \quad (t,x) \in \mathbb{R} \times \Omega,\\ &u(t,x) = f(t,x), \quad (t,x) \in \mathbb{R}_+ \times \partial\Omega,\\ &u(t,x) = 0, \quad (t,x) \in \mathbb{R}_- \times \Omega,\end{aligned} \tag{6}$$

where $S(x, u(t,x))$ is given by (4). Throughout this work, we assume that λ, μ, A, B, C are smooth functions on $\overline{\Omega}$. Here, for simplicity, we took $\rho = 1$. We know (see e.g. [20]) that upon changing variables and introducing lower order terms, the system (5) can always be reduced to $\rho = 1$. Also, we took $F = 0$ in (5). It is easy to see that $u = 0$ is a trivial solution to the problem if $f = 0$. Later, we also use $Z = \mathbb{R} \times \overline{\Omega}$ and $Y = \mathbb{R} \times \partial\Omega$.

The equation in (6) is a second order quasilinear system. In general, the solution may develop shocks and we do not expect long time existence result. We establish the well-posedness for small boundary data in Sect. 2. The novelty of this work is that we analyze the nonlinear interactions of two (distorted) plane waves and show that certain nonlinear responses are generated and they carry the information of the nonlinear parameters. More precisely, let the boundary sources f be

$$f = \epsilon_1 f^{(1)} + \epsilon_2 f^{(2)}$$

depending on two small parameters ϵ_1, ϵ_2. The solution u of (6) with boundary source f has an asymptotic expansion

$$u = \epsilon_1 u^{(1)} + \epsilon_2 u^{(2)} + \epsilon_1^2 u^{(11)} + \epsilon_2^2 u^{(22)} + \epsilon_1\epsilon_2 u^{(12)} + \text{ higher order terms in } \epsilon_1, \epsilon_2.$$

Here, $u^{(1)}, u^{(2)}$ are linear responses satisfying the linearized equations

$$\begin{aligned} Pu^{(\bullet)}(t,x) &= 0, \quad (t,x) \in \mathbb{R} \times \Omega, \\ u^{(\bullet)}(t,x) &= f^{(\bullet)}(t,x), \quad (t,x) \in \mathbb{R}_+ \times \partial\Omega, \\ u^{(\bullet)}(t,x) &= 0, \quad (t,x) \in \mathbb{R}_- \times \Omega, \end{aligned} \tag{7}$$

where $\bullet = 1, 2$ and $u^{(11)}, u^{(12)}, u^{(22)}$ are nonlinear responses satisfying

$$\begin{aligned} Pu^{(ij)}(t,x) &= \nabla \cdot \mathscr{G}(u^{(i)}, u^{(j)}), \quad (t,x) \in \mathbb{R} \times \Omega, \\ u^{(ij)}(t,x) &= 0, \quad (t,x) \in \mathbb{R}_+ \times \partial\Omega, \\ u^{(ij)}(t,x) &= 0, \quad (t,x) \in \mathbb{R}_- \times \Omega, \end{aligned} \tag{8}$$

where $i, j \in \{1, 2\}$ and the term $\mathscr{G}$ is quadratic in $u^{(i)}, u^{(j)}$ and comes from the nonlinear terms of (6), see (22) for its exact form.

The nonlinear interactions of elastic waves are of great interest in seismology, rock sciences etc. In the literatures e.g. [4,7,11] among many others, they have been mostly analyzed by taking $u^{(1)}, u^{(2)}$ as (smooth) plane waves of the form

$$e^{\imath(-tw+\mathbf{k}\cdot x)}\mathbf{a},$$

where $\imath^2 = -1$ and $\mathbf{a}, \mathbf{k} \in \mathbb{R}^3$ are the polarization vector and wave vector respectively. The nonlinear responses are recognized as sum or difference harmonics. One disadvantage of the plane wave approach is that the plane waves extend to the whole space hence it becomes difficult to localize the nonlinear interactions. We shall use distorted plane waves propagating near fixed directions. Locally, they can be expressed as oscillatory integrals of the form

$$\int e^{\imath(t,x)\cdot\xi}\mathbf{a}(t,x;\xi)d\xi,$$

where the amplitude $\mathbf{a}(t,x;\xi)$ belongs to some symbol spaces. The waves and nonlinear responses are characterized using their wave front sets. We construct proper sources $f^{(\bullet)}$ so that $u^{(\bullet)}$ are conormal distributions. This is done in Sect. 3 using microlocal constructions for the initial boundary value problem. The conormal distributions appear frequently in applications, such as Heaviside functions and impulse functions, see [8] for more examples. Next, we show in Theorem 3 that the nonlinear interactions of $u^{(1)}, u^{(2)}$ generates new singularities in $u^{(12)}$. Because of the P–S wave decomposition, there are many cases of the interaction. We are able to determine all the possible responses and find conditions when the responses are non-trivial. The results are summarized in Table 1 in Sect. 4.

1.3 The inverse problem

Our next goal is to determine the elastic parameters from the boundary measurements of the nonlinear responses. We introduce notions to state the result. For the linearized equations

$$Pu = \frac{\partial^2 u}{\partial t^2} - \nabla \cdot \widetilde{S}(x, u) = 0,$$

where $\widetilde{S}(x, u)$ is defined in (2), the characteristic variety of P is the union of subvarieties

$$\begin{aligned} \Sigma_P &= \{(\tau, \xi) \in T^*(\mathbb{R} \times \overline{\Omega}) : \tau^2 - \langle \xi, \xi \rangle_P = 0\}, \quad \langle \xi, \xi \rangle_P = (\lambda(x) + 2\mu(x))|\xi|^2, \\ \Sigma_S &= \{(\tau, \xi) \in T^*(\mathbb{R} \times \overline{\Omega}) : \tau^2 - \langle \xi, \xi \rangle_S = 0\}, \quad \langle \xi, \xi \rangle_S = \mu(x)|\xi|^2, \end{aligned} \tag{9}$$

which corresponds to shear and compressional waves. We assume that

$$\lambda + \mu > 0, \quad \mu > 0 \text{ on } \overline{\Omega}. \tag{10}$$

Then the operator P is a system of real principal type (in the sense of Denker [2]), see [6, Prop. 4.1]. We let $g_{P/S}$ be the Riemannian metric on $\overline{\Omega}$ corresponding to $\langle \cdot \rangle_{P/S}$ and let $\text{diam}_{P/S}(\Omega)$ be the diameter of Ω with respect to $g_{P/S}$. We notice that $\text{diam}_S(\Omega) > \text{diam}_P(\Omega)$ in view of (9) and (10).

Using the well-posedness result established in Sect. 2, we define the displacement-to-traction map as follows. For any fixed $T_0 > 0$, we show in Theorem 2 that there exits $\epsilon_0 > 0$ so that for any $f \in C^m([0, T_0] \times \partial\Omega)$ supported away from $t = 0$ and f sufficiently close to the zero function, there exists a unique solution $u(t, x)$ of (6). Then we define the displacement-to-traction map as

$$\Lambda_{T_0} : f = u|_{[0,T_0]\times\partial\Omega} \to \nu \cdot S(x, u)|_{[0,T_0]\times\partial\Omega},$$

where $\nu = \nu(x)$ is the exterior normal to $\partial\Omega$. We also use Λ for Λ_{T_0} when T_0 is clear from the context.

Theorem 1 *Assume that $\partial\Omega$ is strictly convex with respect to $g_{P/S}$ and there is no conjugate point for $g_{P/S}$ in $\overline{\Omega}$. For $T_0 > 2\text{diam}_S(\Omega)$, the parameters λ, μ, A, B are uniquely determined in $\overline{\Omega}$ by Λ_{T_0}.*

It is worth mentioning that the linear version of Theorem 1 has been extensively studied in the literature. In particular, for the isotropic elastic equations, it is proved in [19] and [6] that the P/S wave speeds (hence the Lamé parameters) are uniquely determined by the displacement-to-traction map. Because the linearized problem in our model is isotropic, the main interest here is to determine the nonlinear parameters. We also remark that our proof leads to an explicit way to reconstruct the nonlinear parameters from the measurement with properly chosen boundary sources. Also, we prove in Proposition 3 that the parameter C cannot be determined at least from the leading term of the generated nonlinear responses. However, it is likely in view of the work [13] that C can be determined from the interaction of three or more waves. This is not pursued in this work.

2 The well-posedness for small boundary data

We establish the well-posedness of the initial boundary value problem (6) which we recall below

$$\begin{aligned}
&\frac{\partial^2 u(t,x)}{\partial t^2} - \nabla \cdot S(x, u(t,x)) = 0, \quad (t,x) \in \mathbb{R} \times \Omega,\\
&u(t,x) = f(t,x), \quad (t,x) \in \mathbb{R}_+ \times \partial\Omega,\\
&u(t,x) = 0, \quad (t,x) \in \mathbb{R}_- \times \Omega,
\end{aligned}$$

where

$$\begin{aligned}
S_{mn}(x,u) = \lambda(x) e_{jj}\left(\delta_{mn} + \frac{\partial u_m}{\partial x_n}\right) + 2\mu(x)\left(e_{nm} + e_{nj}\frac{\partial u_m}{\partial x_j}\right)\\
+ A(x) e_{mj} e_{nj} + B(x)(2 e_{jj} e_{mn} + e_{ij} e_{ij} \delta_{mn}) + C(x) e_{ii} e_{jj} \delta_{mn}.
\end{aligned}$$

In the literature, the well-posedness of quasilinear hyperbolic systems are studied for the initial value problem ($\Omega = \mathbb{R}^3$) in [10] with applications to nonlinear elastodynamics and general relativity. Some variants of the results are obtained by Kato for scalar equations or other initial-boundary conditions. Dafermos and Hrusa studied the initial-boundary value problem for nonlinear elastic equations in [1] which applies to our model. However, only the short time existence result was established for the Dirichlet boundary conditions. Their result is close to what we need. We shall modify their proof to obtain our result. We refer to [16] for similar treatments for one dimensional scalar wave equations.

We denote the L^p based Sobolev space on Ω of order m by $W^{m,p}(\Omega;\mathbb{R})$. The compactly supported Sobolev functions are denoted by $W_0^{m,p}(\Omega,\mathbb{R})$. When $p=2$, we also use $H^m(\Omega) = W^{m,2}(\Omega;\mathbb{R})$, $H_0^m(\Omega) = W_0^{m,2}(\Omega;\mathbb{R})$. For $f \in C^m(M)$, $M \subset \mathbb{R}^4$, we denote the semi-norm by

$$\|f\|_{C^m(M)} \doteq \sup_{x\in M} \sum_{|\alpha|\le m} |\partial_x^\alpha f(x)|.$$

The main result of this section is

Theorem 2 *Let $T_0 > 0$ be fixed. Assume that $f \in C^m([0, T_0] \times \partial\Omega), m \geq 3$ is supported away from $t = 0$. Then there exists $\epsilon_0 > 0$ such that for $\|f\|_{C^m} < \epsilon_0$, there exists a unique solution*

$$u \in \bigcap_{k=0}^{m} C^k([0, T_0]; W^{m-k,2}(\Omega, \mathbb{R}))$$

to (6) *and we have the estimates*

$$\max_{t\in[0,T_0]} \|\partial_t^{m-k} u(t)\|_{W^{m-k,2}(\Omega)} \leq C_0 \|f\|_{C^m(\mathbb{R}\times\partial\Omega)},$$

where $C_0 > 0$ does not depend on f.

We make several remarks. We formulate and prove the result specifically suited to our need. The assumption that f is supported away from $t = 0$ is for simplicity. In general, the theorem should work if f satisfies certain compatibility conditions at $\{0\} \times \partial\Omega$ with the initial conditions. The proof of the theorem is based on some modifications of [1, Theorem 5.2]. Indeed, the proof in [1] is quite involved and was build upon an abstract framework. To minimize the amount of additional work, we will follow [1] very closely, even their notations. We remark that we have not tried to get sharp results which are not necessary for the inverse problem.

Proof of Theorem 2 The first step is to convert the problem to a Dirichlet problem. Suppose that $f \in C^m(\mathbb{R} \times \partial\Omega)$ with $m \geq 3$ and f is compactly supported in $t > 0$. We use the Seeley extension, see [14, Section 1.4]. Following the arguments there, we can find a function $\widetilde{f} \in C^m(\mathbb{R} \times \Omega)$ such that $\widetilde{f}|_{\mathbb{R}\times\partial\Omega} = f$ and $\widetilde{f}$ is supported in $t > 0$. Moreover, the extension is continuous namely,

$$\|\widetilde{f}\|_{C^m(\mathbb{R}\times\Omega)} \leq C\|f\|_{C^m(\mathbb{R}\times\partial\Omega)}.$$

Hereafter, C denotes a generic constant. Let $u = \widetilde{u} + \widetilde{f}$. We have

$$S(u) = \mathscr{A}(\widetilde{u}, \widetilde{f}) + S(\widetilde{f}),$$

where

$$\mathscr{A}_{mn}(\widetilde{u}, \widetilde{f}) = \frac{\partial\mathscr{E}(\widetilde{u} + \widetilde{f})}{\partial(\partial\widetilde{u}_m/\partial x_n)} = S(\widetilde{u}) + \mathscr{I}(t, x, \widetilde{u}, \nabla\widetilde{u}, \widetilde{f}, \nabla\widetilde{f}).$$

Here, $\mathscr{I}$ is a smooth function of its arguments and we recall that $\mathscr{E}$ is the scalar energy function. We can further write

$$\begin{aligned}\left(\nabla \cdot \mathscr{A}(\widetilde{u}, \widetilde{f})\right)_i &= \sum_{j,\alpha,\beta=1}^{3} \widetilde{A}_{i\alpha j\beta} \frac{\partial^2\widetilde{u}_j}{\partial x_\alpha \partial x_\beta} \\ &= (\lambda(x) + \mu(x)) \sum_{j=1}^{3} \frac{\partial^2\widetilde{u}_j}{\partial x_i \partial x_j} + \mu(x) \sum_{j=1}^{3} \frac{\partial^2\widetilde{u}_i}{\partial x_j^2} + E_i(t, x, \widetilde{u}, \nabla\widetilde{u}, \widetilde{f}, \nabla\widetilde{f}),\end{aligned}$$

where E_i denotes the nonlinear terms. Because $\widetilde{A}_{i\alpha j\beta}$ comes from a scalar energy function, we know (see e.g. [1, Section 1]) that $\widetilde{A}_{i\alpha j\beta} = \widetilde{A}_{i\beta j\alpha}$ are symmetric. Moreover, because of the assumptions on λ, μ and the compactness of $\overline{\Omega}$, $\widetilde{A}$ satisfy the strong ellipticity condition, namely there exists $\delta > 0$ such that

$$\widetilde{A}_{i\alpha j\beta}\xi_i\xi_j\zeta_\alpha\zeta_\beta \geq \delta|\xi|^2|\zeta|^2, \quad \xi, \zeta \in \mathbb{R}^3,$$

for all $\widetilde{u}$ in a sufficiently small open neighborhood $\mathscr{O}$ of the zero function in $C^m(\mathbb{R}\times\overline{\Omega})$ such that $\det(I + \nabla\widetilde{u}) > 0$.

Now $\widetilde{u} = u - \widetilde{f}$ satisfies the equation

$$\begin{aligned} &\frac{\partial^2\widetilde{u}}{\partial t^2} - \nabla\cdot\mathscr{A}(\widetilde{u}, \widetilde{f}) = \mathscr{F}, \quad \text{in } \mathbb{R}\times\overline{\Omega}, \\ &\widetilde{u} = 0, \quad \text{in } (\mathbb{R}_+\times\partial\Omega)\cup(\mathbb{R}_-\times\overline{\Omega}), \end{aligned} \tag{11}$$

where

$$\mathscr{F} = S(\widetilde{f}) - \frac{\partial^2\widetilde{f}}{\partial t^2} \in C^{m-2}([0, T]\times\overline{\Omega}).$$

It is clear that

$$\|\mathscr{F}\|_{C^{m-2}} \leq C\|\widetilde{f}\|_{C^m} \leq C\|f\|_{C^m}.$$

Then the assumptions of [1, Theorem 5.2] are all satisfied and the problem can be reduced to the following abstract problem studied in [1, Section 4]: for any $T_0 > 0$, consider the initial value problem

$$\begin{aligned} &\frac{\partial^2 u}{\partial t^2} + E(t, x, u, \partial u, f, \partial f)u = F, \quad \text{in } [0, T_0]\times\overline{\Omega}, \\ &u = 0, \quad \text{in } \mathbb{R}_-\times\overline{\Omega}, \end{aligned} \tag{12}$$

where E satisfies the assumptions (E1)–(E4) and F (automatically) satisfies assumptions (g1)–(g2) of [1, Section 4]. Here, to conform with the notations in [1], we have changed the meaning of u and f so that they are C^m functions on $\mathbb{R}\times\overline{\Omega}$. Also, we let $H_s = W^{s,2}(\Omega, \mathbb{R})$, $V = W_0^{1,2}(\Omega, \mathbb{R})$ and $X_s = V\cap H_s$. In [1, Theorem 4.2], the local in time existence was established for this abstract problem. In particular, for any $F \in C^m([0, T]\times\overline{\Omega})$, there exists a $T_0 > 0$ and a unique solution

$$u \in \bigcap_{k=0}^{m} C^{m-k}([0, T_0]; X_{m-k}).$$

The proof of Theorem 5.2 of [1] follows from this result. Here, we claim that for fixed $T_0 > 0$, if F is sufficiently small, there exists a unique solution u of (12) as above and $\|u\|_{C^m} \leq C\|F\|_{C^m}$. The proof of the claim is based on modifying that of Theorem

4.2 of [1], which is essentially built upon Theorem 4.1 of [1] for a simplified version of the problem (12) i.e.

$$\frac{\partial^2 u}{\partial t^2} + A(u)u = F, \quad \text{in } [0, T_0] \times \overline{\Omega},$$
$$u = 0, \quad \text{in } \mathbb{R}_- \times \overline{\Omega}, \tag{13}$$

where A satisfies the assumptions in Section 4 of [1]. To clearly indicate the modifications we need, we shall prove our claim for this problem.

For $M, T > 0$, we define a function space $Z(M, T)$ consisting of all functions w satisfying

$$w \in \bigcap_{k=1}^{m} W^{k,\infty}([0, T]; H_{m-k}), \quad \text{ess-sup}_{t\in[0,T]} \sum_{k=0}^{3} \|w(t)\|_{m-k}^2 \leq M^2.$$

For $w \in Z(M, T_0)$, consider the linearized problem

$$\frac{\partial^2 u}{\partial t^2} + A(t, x, w)u = F, \quad \text{in } [0, T_0] \times \Omega,$$
$$u = 0, \quad \text{in } \mathbb{R}_- \times \overline{\Omega}. \tag{14}$$

For this problem, Theorem 3.1 of [1] shows that there exists a unique solution $u \in \bigcap_{k=1}^{m} W^{k,\infty}([0, T_0]; H_{m-k})$ with the estimate

$$\sum_{k=0}^{m} \|u(t)\|_{m-k}^2 \leq C_0 N(T_0) e^{K_0 T_0}, \quad t \in [0, T_0],$$

where C_0, K_0 are positive constants depending only on the coefficient of the equation, and

$$N(T_0) = \sup_{t\in[0,T_0]} \sum_{k=0}^{m-2} \|F(t)\|_{m-2-k}^2.$$

We observe that $N(T_0) = O(\epsilon^2)$ if $\|F\|_{C^m} \leq \epsilon$. We denote by $\mathscr{T}$ the map which maps $w \in Z(M, T)$ to the solution of (14). We let

$$M_0^2 = 4N(T_0)C_0 e^{K_0 T_0} = O(\epsilon)$$

and choose ϵ sufficiently small so that $\mathscr{T}$ maps $Z(M_0, T_0)$ into itself. Following the rest of proof of [1, Theorem 4.1] especially equation (4.37), we see that the map is a contraction if

$$C M_0 T_0 e^{C M_0 T_0} < 1.$$

We choose ϵ_0 sufficiently small so this is true. This finishes the proof of the claim. This implies the claim for system (12) which further concludes the proof of the theorem.

3 Microlocal analysis of the linearized system

We consider the initial boundary value problem for the linearized equation (7) recalled below

$$
\begin{aligned}
Pu(t,x) = \frac{\partial^2 u(t,x)}{\partial t^2} - \nabla \cdot \widetilde{S}(x, u(t,x)) = 0, \quad (t,x) \in \mathbb{R} \times \Omega,\\
u(t,x) = f(t,x), \quad (t,x) \in \mathbb{R}_+ \times \partial\Omega,\\
u(t,x) = 0, \quad (t,x) \in \mathbb{R}_- \times \Omega.
\end{aligned} \tag{15}
$$

where

$$
\widetilde{S}_{mn}(x,u) = \lambda(x) \frac{\partial u_j}{\partial x_j} \delta_{mn} + \mu(x) \left(\frac{\partial u_m}{\partial x_n} + \frac{\partial u_n}{\partial x_m} \right).
$$

Our goal is to construct boundary sources f so that the solution u has conormal type singularities propagating into the region Ω. Such u will be called *distorted plane waves*. We start with basic microlocal analysis for boundary value problems of the linear system.

Let $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$ be the dual coordinate of x in $T^*_x\overline{\Omega}$ and we let $(t, x; \tau, \xi) \in T^*(\mathbb{R} \times \overline{\Omega})\backslash 0$ be the local coordinates. The Euclidean metric of $\mathbb{R}^3$ is used to define inner product $\xi \cdot \xi$, $\forall \xi \in \mathbb{R}^3$ and to identify tangent and co-tangent vectors on $\mathbb{R}^3$. For a non-zero direction $\xi \in \mathbb{R}^3\backslash 0$, we denote by $\pi = \pi(\xi) = \xi \otimes \xi / (\xi \cdot \xi)$ the orthogonal projection to ξ. From [6, Proposition 4.1], we know that P is a system of real principal type (in the sense of Dencker [2]) with principal symbol

$$
\mathbf{p} = \mathbf{p}_S(\mathrm{Id} - \pi) + \mathbf{p}_P \pi,
$$

where

$$
\mathbf{p}_{P/S}(t, x, \tau, \xi) = \tau^2 - \langle \xi, \xi \rangle_{P/S}.
$$

For $\mu > 0$, we see from (9) that $0 < \langle \xi, \xi \rangle_S < \langle \xi, \xi \rangle_P$, $\xi \in \mathbb{R}^3\backslash 0$. It is well-known that the system P can be decoupled as follows. We decompose u to the P/S modes as

$$
u^P = \Pi_P u = \Delta^{-1} \nabla(\nabla \cdot u) \quad \text{and} \quad u^S = \Pi_S u = (\mathrm{Id} - \Pi_P) u,
$$

where $\nabla = (\partial_{x_1}, \partial_{x_2}, \partial_{x_3})$ is the gradient and $\Delta = \sum_{i=1}^3 \partial_{x_i}^2$ is the Laplacian. Observe that the symbols of Π_P, Π_S are $\sigma(\Pi_P)(x, \xi) = \pi(\xi)$ and $\sigma(\Pi_S)(x, \xi) = \mathrm{Id} - \pi(\xi)$, $(x, \xi) \in T^*\mathbb{R}^3$. It follows from Taylor's diagonalization method [21] (see also [20, Lemma 2.1]) that $Pu = 0$ is equivalent to

$$\frac{\partial^2 u^P}{\partial t^2} = [(\lambda + 2\mu)\Delta + B_1]u^P + R_1 u,$$
$$\frac{\partial^2 u^S}{\partial t^2} = [\mu \Delta + B_2]u^S + R_2 u, \tag{16}$$

where B_1, B_2 are first order pseudo differential operators (denoted by $\Psi^1(\mathbb{R}^3)$) and R_1, R_2 are smoothing operators. The boundary data f can be decomposed to $f = f^P + f^S$ so the system (7) is decoupled up to a smoothing term.

For the two symbols $\mathbf{p}_{P/S}$, the corresponding Hamiltonian vector fields are

$$H_{\mathbf{p}_{P/S}} = -2\tau \frac{\partial}{\partial t} + \sum_{i=1}^{3} \left[\frac{\partial \langle \xi, \xi \rangle_{P/S}}{\partial \xi_i} \frac{\partial}{\partial x_i} - \frac{\partial \langle \xi, \xi \rangle_{P/S}}{\partial x_i} \frac{\partial}{\partial \xi_i} \right].$$

The integral curves on $T^*(\mathbb{R} \times \overline{\Omega})$ are called bicharacteristics. For $x \in \partial\Omega$, $\xi \in T_x^*\overline{\Omega}$, we define the projection $\pi_\partial : T_x^*(\mathbb{R} \times \overline{\Omega}) \to T_x^*(\mathbb{R} \times \partial\Omega)$ by $\pi_\partial(\xi) = \xi|_{T_x^*(\partial\Omega)}$. The point $\gamma = (t, x; \tau, \pi_\partial(\xi)) \in T^*(\mathbb{R} \times \partial\Omega)\backslash 0$ is called *elliptic, hyperbolic or glancing* for P/S mode if the following quadratic equation in z

$$\mathbf{p}_{P/S}(t, x; \tau, \xi - z\nu(x)) = 0$$

has no real roots, two distinct real roots or a double real roots, see [6, Section 4]. The cotangent bundle $T^*(\mathbb{R} \times \partial\Omega)$ is decomposed into elliptic regions $\mathscr{E}_{P/S}$, hyperbolic regions $\mathscr{H}_{P/S}$ and the glancing hypersurfaces $\mathscr{G}_{P/S}$ for the P/S modes. Because of the assumption that $\mu > 0$, it is easy to see that $\mathscr{E}_S \subset \mathscr{E}_P$ and $\mathscr{H}_P \subset \mathscr{H}_S$. We let $\mathscr{G} = \mathscr{G}_P \cup \mathscr{G}_S$. A simple real root z is called forward (backward) if the bicharacteristic curve starting in direction $\xi - z\nu$ enters $\mathbb{R} \times \Omega$ when time increases (decreases). We denote by $z_{P/S}$ the forward real root or the complex root z with positive imaginary part of $\mathbf{p}_{P/S}(t, x, \tau, \xi - z\nu) = 0$, and we use $\xi_{P/S} = \xi - z_{P/S}\nu(x)$.

Consider the displacement-to-traction map of the linear system (7), that is $\Lambda_{lin}(f) = \nu \cdot \widetilde{S}(u)$. We will see later in (18) that this is just the linearization of the displacement-to-traction map for the nonlinear system (11). It is proved in [6, Proposition 4.2] that Λ_{lin} is a first order pseudo-differential operator near every non-glancing point $\gamma \in T^*(\mathbb{R} \times \partial\Omega)\backslash\mathscr{G}$.

For a Lagrangian submanifold Λ of T^*M e.g. $M = \mathbb{R} \times \Omega$, the Lagrangian distributions of order μ are denoted by $I^\mu(\Lambda)$, see [9] for the definition. Let K be a codimension k submanifold of M. The conormal bundle $N^*K = \{(x, \zeta) \in T^*M\backslash 0 : x \in K, \zeta|_{T_x K} = 0\}$ is a Lagrangian submanifold. The conormal distributions of order μ to K are denoted by $I^\mu(N^*K)$.

Now let K be a codimension one submanifold of $\mathbb{R} \times \partial\Omega$ (hence codimension two in $\mathbb{R} \times \overline{\Omega}$). We use N_∂^*K to denote the conormal bundle of K as a submanifold of the boundary $\mathbb{R} \times \partial\Omega$ and N^*K the conormal bundle in $\mathbb{R} \times \overline{\Omega}$. We assume that $N_\partial^*K \cap \mathscr{H}_P$ has an open interior and consider distributions $f \in I^\mu(N_\partial^*K)$. Indeed, we are interested in the singularities of f in the hyperbolic directions. We introduce

$$\Lambda_K^P = (\mathbb{R} \times \overline{\Omega}) \cap \left(\bigcup_{s \geq 0} \exp s H_{\mathbf{p}_P} (N^* K \cap \Sigma_P) \right),$$

$$\Lambda_K^S = (\mathbb{R} \times \overline{\Omega}) \cap \left(\bigcup_{s \geq 0} \exp s H_{\mathbf{p}_S} (N^* K \cap \Sigma_S) \right).$$

These are Lagrangian submanifolds of $T^*(\mathbb{R} \times \overline{\Omega})$. Their projections to $\mathbb{R} \times \overline{\Omega}$ are geodesic flow out of N^*K with respect to the Lorentzian metrics $-dt^2 + g_{P/S}$.

Proposition 1 *Let K, f be defined as above and u be the solution of (7) with boundary source f. Let $f = f^P + f^S$ and $u = u^P + u^S$. We have the following conclusions.*

1. *There exists (Fourier integral) operators $Q_{bdy}^{P/S}$ such that $u^{P/S} = Q_{bdy}^{P/S}(f^{P/S}) \in I^{\mu-1/4}(\Lambda_K^{P/S})$ are Lagrangian distributions.*
2. *Let $(z, \zeta) \in T^*(\mathbb{R} \times \overline{\Omega})$ lie on the bicharacteristic strip of $H_{\mathbf{p}_{P/S}}$ from $(z_0, \zeta_{0,P/S})$ for some $(z_0, \zeta_0) \in T^*(\mathbb{R} \times \partial\Omega)$. Then the principal symbols of $u^{P/S}$ and $f^{P/S}$ are related by*

$$\sigma(u^{P/S})(z, \zeta) = Q_{bdy}^{P/S}(z, \zeta, z_0, \zeta_0)\sigma(f^{P/S})(z_0, \zeta_0),$$

where $Q_{bdy}^{P/S}$ are 3×3 invertible matrices and bdy stands for boundary value problem.

Proof For simplicity, we use $Z = \mathbb{R} \times \overline{\Omega}$ and $Y = \mathbb{R} \times \partial\Omega$. Locally near Y, we can make a change of variable to flat the boundary. Then the problem (7) is equivalent to the Cauchy problem for the second order system $Pu \in C^\infty(Z)$ with Cauchy data

$$Cu = (\rho_0 u, \rho_0 \nabla \cdot \tilde{S}(u)) = (f, \Lambda_{lin}(f)),$$

where ρ_0 is the restriction operator to Y. In particular, ρ_0 is an Fourier integral operator in $I^{1/4}(Z, Y; R_0)$, where the canonical relation

$$R_0 = \{((z_0, \zeta_0), (z, \zeta)) \in (T^*Y \times T^*Z) \backslash 0 : z_0 = z, \zeta_0 = \pi_\partial(\zeta) = \zeta|_{T^*_{z_0}Y}\},$$

see [3, Section 5.1]. According to [3, Theorem 5.2.1], there exist Fourier integral operators $Q_0 \in I^{-1/4}(Z, Y; C_0)$ and $Q_1 \in I^{-1-1/4}(Z, Y; C_0)$ which are maps $\mathscr{E}'(Y) \to \mathscr{D}'(Z)$ such that

$$PQ_i \in C^\infty(Z), \quad \rho_0 Q_j = \delta_{0j}, \quad \rho_0 \Lambda Q_j = \delta_{1j}, \qquad i, j = 0, 1,$$

where C_0 is the canonical relation

$\{((z, \zeta), (z_0, \zeta_0)) : (z, \zeta) \in T^*Z$ is on the bicharacteristic strip of $\mathbf{p}$ through some $(z_0, \hat{\zeta}) \in T^*Z$ such that $\pi_\partial(\hat{\zeta}) = \zeta_0$, for $(z_0, \zeta_0) \in T^*Y\}$.

Suppose that $f \in I^{\mu}(N^*_\partial K)$ is conormal. By the composition of Fourier integral operators (see e.g. [9]), we have $Q_0 f \in I^{\mu-1/4}(\Lambda_K)$, $Q_1 f \in I^{\mu-1-1/4}(\Lambda_K)$. So the solution $u = Q_0 f + Q_1 f \in I^{\mu-1/4}(\Lambda_K)$. Suppose that $((z,\zeta),(z_0,\zeta_0)) \in C_0$, then the principal symbol

$$\sigma(u)(z,\zeta) = \sigma(Q_0)(z,\zeta,z_0,\zeta_0)\sigma(f)(z_0,\zeta_0),$$

where Q_0 is invertible. Finally, we apply these arguments to the decoupled system and let $Q^{P/S}_{bdy} = Q_0^{P/S} + Q_1^{P/S} \in I^{-1/4}(Z, Y; C_0^{P/S})$ where

$$\begin{aligned} C_0^{P/S} = \{((z,\zeta),(z_0,\zeta_0)) : (z,\zeta) \in T^*Z \cap \Sigma_{P/S} \text{ is on the bicharacteristic strip of} \\ \mathbf{p}_{P/S} \text{ through some } (z_0,\hat{\zeta}) \in T^*Z \cap \Sigma_{P/S} \text{ such that } \pi_\partial(\hat{\zeta}) = \zeta_0, \text{ for } (z_0,\zeta_0) \in T^*Y\}. \end{aligned}$$

This completes the proof.

At last, we use the proposition to construct distorted plane waves. Let $\gamma_0 = (t_0, x_0, \tau_0, \xi_0) \in T^*(\mathbb{R} \times \partial\Omega)\backslash\mathscr{G}, t_0 > 0, x_0 \in \partial\Omega$ be a hyperbolic point in $\mathscr{H}_P \subset \mathscr{H}_S$. We let K_0 be a codimension one submanifold of $\mathbb{R} \times \partial\Omega$ so that $\gamma_0 \in N^*_\partial K_0$. For $\delta > 0$ sufficiently small, we define

$$K(\gamma_0;\delta) = K_0 \cap \{(t,x) \in \mathbb{R} \times \partial\Omega : |t - t_0| < \delta, \operatorname{dist}(x, x_0) < \delta\},$$

which is a small neighborhood of (t_0, x_0) contained in K_0. Then $\Gamma_0(\delta) \doteq N^*_\partial K(\gamma_0;\delta)$ is a small open neighborhood of γ_0 and $\Gamma_0(\delta) \cap \mathscr{H}_P \neq \emptyset$. As $\delta \to 0$, the set $\Gamma_0(\delta)$ tends to the vector γ_0. Now we consider their flow out under the Hamilton vector fields of $\mathbf{p}_{P/S}$

$$\begin{aligned} \Lambda^P(\gamma_0;\delta) &= \bigcup_{s\geq 0} \exp sH_{\mathbf{p}_P}(N^*K(\gamma_0;\delta) \cap \Sigma_P), \\ \Lambda^S(\gamma_0;\delta) &= \bigcup_{s\geq 0} \exp sH_{\mathbf{p}_S}(N^*K(\gamma_0;\delta) \cap \Sigma_S). \end{aligned}$$

which are Lagrangian submanifolds of $T^*(\mathbb{R} \times \overline{\Omega})$. As $\delta \to 0$, they tend to the forward bicharacteristics corresponding to $\gamma_0^{P/S} = (t_0, x_0, \tau_0, \xi_{0,P/S})$. By our non-conjugate point assumption, we know that the projections of $\Lambda_0^{P/S}$ to $\mathbb{R} \times \overline{\Omega}$ should be co-dimension one submanifolds $\mathscr{P}_0, \mathscr{S}_0$. So we have

$$\Lambda^P(\gamma_0;\delta) = N^*\mathscr{P}_0, \quad \Lambda^S(\gamma_0;\delta) = N^*\mathscr{S}_0.$$

Also, as $\delta \to 0$, $\mathscr{P}_0$ tends to the geodesic of the metric $-dt^2 + g_P$ from γ_0^P and $\mathscr{S}_0$ tends to the geodesic of the metric $-dt^2 + g_S$ from γ_0^S. For $f \in I^{\mu+1/4}(N^*_\partial K(\gamma_0;\delta))$, the solution u of (7) satisfies $u = u^P + u^S, u^P \in I^{\mu}(N^*\mathscr{P}_0), u^S \in I^{\mu}(N^*\mathscr{S}_0)$, which is called a *distorted plane wave*. See Fig. 1. We see that for δ small, the singular supports of $u^{P/S}$ are close to the corresponding geodesics from $\gamma_0^{P/S}$.

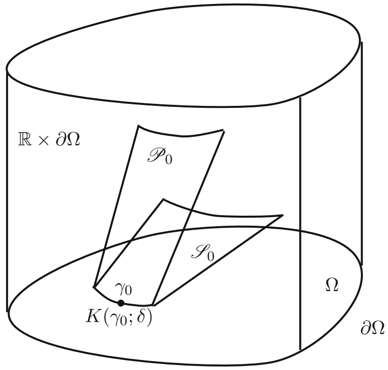


Fig. 1 Construction of distorted plane waves

4 The nonlinear interaction

4.1 Construction of sources

We consider the nonlinear effects in this section. First, we construct two distorted plane waves.

Definition 1 Let $\gamma_1, \gamma_2 \in T^*(\mathbb{R}_+ \times \partial\Omega)\backslash\mathscr{G}$ be hyperbolic points and construct two sources $f^{(1)} \in I^{\mu+1/4}(N_\partial^* K(\gamma_1; \delta))$ and $f^{(2)} \in I^{\mu+1/4}(N_\partial^* K(\gamma_2; \delta))$ with $\mu < -23/4$ as in the end of Sect. 3. The corresponding distorted plane waves are denoted by $u^{(1)}, u^{(2)}$. We write

$$u^{(\bullet)} = u^{(\bullet),S} + u^{(\bullet),P}, \quad \bullet = 1, 2,$$

such that

$$u^{(\bullet),P} \in I^{\mu}(\Lambda^P(\gamma_\bullet; \delta)) = I^{\mu}(N^*\mathscr{P}_\bullet),$$
$$u^{(\bullet),S} \in I^{\mu}(\Lambda^S(\gamma_\bullet; \delta)) = I^{\mu}(N^*\mathscr{S}_\bullet).$$

Here, $\mathscr{P}_\bullet, \mathscr{S}_\bullet$ are codimension one submanifolds of $\mathbb{R}\times\overline{\Omega}$. We assume that $\mathscr{P}_i \cap \mathscr{S}_i = \emptyset$, $i = 1, 2$, (i.e. no self-interactions) and that

$$\mathscr{P}_1 \cap \mathscr{P}_2 = \mathscr{Z}_{PP}, \quad \mathscr{S}_1 \cap \mathscr{S}_2 = \mathscr{Z}_{SS}, \quad \mathscr{P}_1 \cap \mathscr{S}_2 = \mathscr{Z}_{PS}, \quad \mathscr{S}_1 \cap \mathscr{P}_2 = \mathscr{Z}_{SP},$$

where the above intersections are either empty or transversal so the $\mathscr{Z}_\bullet$ are codimension two submanifolds.

We would like to construct a source $f = \epsilon_1 f^{(1)} + \epsilon_2 f^{(2)}$ for two small parameters $\epsilon_1, \epsilon_2 > 0$ so that the linearized solutions are distorted plane waves. In general, this might lead to reflections of the waves at the boundary and it becomes difficult to determine the nonlinear responses. Therefore, we proceed as follows.

Proposition 2 *For $f^{(1)}, f^{(2)}, u^{(1)}, u^{(2)}$ in Definition 1 and ϵ_1, ϵ_2 sufficiently small, there exists $f_\epsilon \in C^2(\mathbb{R} \times \partial\Omega)$ supported in $\mathbb{R}_+ \times \partial\Omega$ so that the solution u_ϵ of (6) has the expansion*

$$u_\epsilon = \epsilon_1 u^{(1)} + \epsilon_2 u^{(2)} + \epsilon_1^2 u^{(11)} + \epsilon_2^2 u^{(22)} + \epsilon_1\epsilon_2 u^{(12)} + o(\epsilon_1^2) + o(\epsilon_2^2), \tag{17}$$

where $u^{(\bullet)}, \bullet = 11, 12, 22$ are determined by $u^{(1)}, u^{(2)}$ through (8).

In the expansion (17), we let $v = \epsilon_1 u^{(1)} + \epsilon_2 u^{(2)}$ and call it the *linear response*. The terms $u^{(11)}, u^{(22)}$ and $u^{(12)}$ are called *nonlinear responses*. We are particularly interested in $u^{(12)}$ as we shall show below that it contains new singularities which do not belong to the linear response. The point of the proposition is revealed in the displacement-to-traction map Λ. We have

$$\partial_{\epsilon_i}\Lambda(f_\epsilon)|_{\epsilon_i=0} = \nu\cdot\widetilde{S}(u^{(i)})|_{\mathbb{R}\times\partial\Omega} = \Lambda_{lin}(f^{(i)}), \quad i = 1, 2, \tag{18}$$

and

$$\partial_{\epsilon_1}\partial_{\epsilon_2}\Lambda(f_\epsilon)|_{\epsilon_1=\epsilon_2=0} = \nu\cdot\widetilde{S}(u^{(12)})|_{\mathbb{R}\times\partial\Omega} + \nu\cdot(\nabla\cdot\mathscr{G}(u^{(1)}, u^{(2)}))|_{\mathbb{R}\times\partial\Omega}, \tag{19}$$

where $\mathscr{G}(\cdot,\cdot)$ is the quadratic term in (8), see also (22).

Proof of Proposition 2 For $\epsilon_1, \epsilon_2 > 0$, we take

$$f_\epsilon = \epsilon_1 u^{(1)}|_{\mathbb{R}\times\partial\Omega} + \epsilon_2 u^{(2)}|_{\mathbb{R}\times\partial\Omega} + f_{\epsilon^2},$$

where $\widetilde{f}_{\epsilon^2}$ consists of higher order terms in ϵ_1, ϵ_2 and is to be specified below. From the finite speed of propagation for the linear system, we see that $f_\epsilon = \epsilon_1 f^{(1)} + \epsilon_2 f^{(2)}$ modulo higher order terms in a sufficiently small neighborhood of γ_1, γ_2. Now consider the regularity. Recall that for a codimension k submanifold K of M of dimension n, we have

$$I^\mu(N^*K) \subset H^s(M) \subset C^r(M), \tag{20}$$

where $s < -\mu - n/4$ and $r < s - n/2$. We should take $\mu < -9/2$ so that $f^{(1)}, f^{(2)} \in C^2(\mathbb{R}\times\partial\Omega)$. We apply Theorem 2. For any $T_0 > 0$, there exists $\epsilon_0 > 0$ such that for $\epsilon_1, \epsilon_2 < \epsilon_0$, there exists a unique solution u_ϵ of (11) with boundary source f_ϵ such that

$$u_\epsilon \in E^2(\mathbb{R}\times\overline{\Omega}) \doteq \bigcap_{k=0}^{2} C^k([0, T_0]; H^{2-k}(\overline{\Omega})) \subset H^2(\mathbb{R}\times\overline{\Omega})$$

and we have the asymptotic expansion (17) in which the remainder terms are also in $E^2(\mathbb{R}\times\overline{\Omega})$. Now we need more regularity so that $u_\epsilon \in C^2(\mathbb{R}\times\overline{\Omega})$. Thus, we demand that in (20) $s = 5$ and $\mu < -5 - 3/4 = -23/4$. Then we let

$$f_{\epsilon^2} = [\epsilon_1^2 u^{(11)} + \epsilon_2^2 u^{(22)} + \epsilon_1\epsilon_2 u^{(12)}]|_{\mathbb{R}\times\partial\Omega} + f_{\epsilon^3},$$

where $f_{\epsilon^3} \in C^2(\mathbb{R}\times\partial\Omega)$ and $f_{\epsilon^3} = o(\epsilon_1^2) + o(\epsilon_2^2)$. We see that f_{ϵ^3} will not affect the terms in the asymptotic expansion (17). This finishes the proof.

We remark that since we will only concern γ_1, γ_2 so the corresponding bicharacteristics do not meet at the boundary, the wave front of $\nu \cdot (\nabla \cdot \mathscr{G}(u^{(1)}, u^{(2)}))|_{\mathbb{R}\times\partial\Omega}$ in (19) will be contained in that of $u^{(1)}$ and $u^{(2)}$. Thus it suffices to find the singularities of $\nu \cdot \widetilde{S}(u^{(12)})|_{\mathbb{R}\times\partial\Omega}$ in $u^{(12)}$ which we do next.

4.2 Generation of the nonlinear response

Among all the nonlinear terms in (6), we only consider the quadratic terms in $S(u)$, denoted by $G(u,u)$ where

$$\begin{aligned} G_{mn}(u,w) = {} & \lambda\widetilde{e}_{jj}\frac{\partial w_m}{\partial x_n} + \frac{1}{2}\lambda\left(\frac{\partial u_k}{\partial x_j}\right)\left(\frac{\partial w_k}{\partial x_j}\right)\delta_{mn} + 2\mu\widetilde{e}_{nj}\frac{\partial w_m}{\partial x_j} + \mu\frac{\partial u_k}{\partial x_m}\frac{\partial w_k}{\partial x_n} \\ & + A\widetilde{e}_{mj}\widetilde{f}_{nj} + B(2\widetilde{e}_{jj}\widetilde{f}_{mn} + \widetilde{e}_{ij}\widetilde{f}_{ij}\delta_{mn}) + C\widetilde{e}_{ii}\widetilde{f}_{jj}\delta_{mn}, \end{aligned} \tag{21}$$

where $\widetilde{f}_{mn} = \frac{1}{2}\left(\frac{\partial w_m}{\partial x_n} + \frac{\partial w_n}{\partial x_m}\right)$. Because $G(u,w)$ is not symmetric, we let

$$\mathscr{G}(u,w) = G(u,w) + G(w,u).$$

Then we see that for $v = \epsilon_1 u^{(1)} + \epsilon_2 u^{(2)}$,

$$G(v,v) = \epsilon_1^2\frac{1}{2}\mathscr{G}(u^{(1)},u^{(1)}) + \epsilon_2^2\frac{1}{2}\mathscr{G}(u^{(2)},u^{(2)}) + \epsilon_1\epsilon_2\mathscr{G}(u^{(1)},u^{(2)}).$$

Thus $u^{(12)}$ is the solution of

$$Pu^{(12)} = \frac{\partial^2 u^{(12)}}{\partial t^2} - \nabla\cdot\widetilde{S}(u^{(12)}) = \nabla\cdot\mathscr{G}(u^{(1)},u^{(2)}) \tag{22}$$

with zero initial and boundary conditions. This is the precise form of the Eq. (8). If we choose the parameter δ in the distorted plane waves sufficiently small, $\mathscr{G}(u^{(1)}, u^{(2)})$ is compactly supported in $\mathbb{R}_+ \times \overline{\Omega}$. Thus by the finite speed of propagation, we can treat (22) as a source problem on $\mathbb{R}\times\mathbb{R}^3$ before the waves reaches the boundary. Although this is not necessary for our proof, it is worth mentioning that in [18] Rachele showed the determination of λ, μ and their normal derivatives to any order on the boundary $\mathbb{R}\times\partial\Omega$ from Λ_{lin}. Thus one can ignore the boundary and extend the system (22) to $\mathbb{R}\times\mathbb{R}^3$. Because of the P/S decomposition, we have

$$\begin{aligned} & \mathscr{G}(u^{(1)},u^{(2)}) \\ & = \mathscr{G}(u^{(1),P},u^{(2),P}) + \mathscr{G}(u^{(1),P},u^{(2),S}) + \mathscr{G}(u^{(1),S},u^{(2),P}) + \mathscr{G}(u^{(1),S},u^{(2),S}) \\ & = \mathscr{G}^{PP} + \mathscr{G}^{PS} + \mathscr{G}^{SP} + \mathscr{G}^{SS}, \end{aligned}$$

where the $\mathscr{G}^\bullet$ in the second line corresponds to the four terms in the first line. These terms represent the P–P interactions, P–S interactions, S–P interactions and S–S interactions. Their singularities can be described using the notion of paired Lagrangian distributions. Let M be an n-dimensional smooth manifold. For two Lagrangians $\Lambda_0, \Lambda_1 \subset T^*M$ intersecting cleanly at a co-dimension k submanifold i.e. $T_q\Lambda_0 \cap T_q\Lambda_1 = T_q(\Lambda_0 \cap \Lambda_1), \ \forall q \in \Lambda_0 \cap \Lambda_1$, the paired Lagrangian distribution associated with (Λ_0, Λ_1) is denoted by $I^{p,l}(\Lambda_0, \Lambda_1)$. The wave front sets of such distributions are contained in $\Lambda_0 \cup \Lambda_1$. We refer the reader to [5,15] for the precise definition and properties.

Now consider $\mathscr{G}_{PP}$ and assume $\mathscr{Z}_{PP} \neq \emptyset$. From [22, Lemma 4.1], we know that the components of $\nabla u^{(\bullet),P}$ are in $I^{\mu+1}(\Lambda_\bullet^P)$, $\bullet = 1, 2$. Then we can apply [5, Lemma 2.1] to get

$$\mathscr{G}_{PP} \in I^{\mu+1,\mu+2}(N^*\mathscr{Z}_{PP}, N^*\mathscr{P}_1) + I^{\mu+1,\mu+2}(N^*\mathscr{Z}_{PP}, N^*\mathscr{P}_2).$$

Using [22, Lemma 4.1] again, we get

$$\nabla \cdot \mathscr{G}_{PP} \in I^{\mu+2,\mu+2}(N^*\mathscr{Z}_{PP}, N^*\mathscr{P}_1) + I^{\mu+2,\mu+2}(N^*\mathscr{Z}_{PP}, N^*\mathscr{P}_2).$$

The wave front set of $\mathscr{G}^{PP}$ is contained in the union of $N^*\mathscr{Z}_{PP}$ and $N^*\mathscr{P}_1, N^*\mathscr{P}_2$. For the propagation of the nonlinear response, we are interested in the co-vectors of $N^*\mathscr{Z}_{PP}$ which are also in Σ_P or Σ_S.

Lemma 1 *Suppose that $\mathscr{P}_1$ intersect $\mathscr{P}_2$ transversally at $\mathscr{Z}_{PP} \neq \emptyset$. Then*

1. $(N^*\mathscr{Z}_{PP} \backslash (N^*\mathscr{P}_1 \cup N^*\mathscr{P}_2)) \cap \Sigma_P = \emptyset$.
2. *For any $p \in \mathscr{Z}_{PP}$, there are two linearly independent vectors $\zeta_+, \zeta_- \in \Sigma_S \cap N^*\mathscr{Z}_{PP}$ at p.*

Proof We remark that (1) is a known fact, but we give an elementary proof below for completeness. Let $p = (t, x) \in \mathscr{Z}_{PP}$ and $\zeta^{(1)} \in N_p^*\mathscr{P}_1, \zeta^{(2)} \in N_p^*\mathscr{P}_2$. We write $\zeta^{(i)} = (\tau^i, \xi^i), \tau^i \in \mathbb{R}, \xi^i \in \mathbb{R}^3, i = 1, 2$. Then we have

$$(\tau^i)^2 = (\lambda + 2\mu)|\xi^i|^2, \quad i = 1, 2.$$

Now consider vectors $\zeta = a\zeta^{(1)} + b\zeta^{(2)} \in N^*\mathscr{Z}_{PP}, a, b \in \mathbb{R}$. Without loss of generality, we assume that $a \neq 0$ and rescale the vectors so that $|\xi^{(\bullet)}| = 1$ and $a = 1$. If $\zeta \in \Sigma_P$, we have

$$(\tau^1 + b\tau^2)^2 = (\lambda + 2\mu)|\xi^1 + b\xi^2|^2 \Rightarrow b(1 - \xi^1 \cdot \xi^2) = 0.$$

Because $\xi^1 \cdot \xi^2 \neq 0$, we conclude that $b = 0$ which implies $\zeta = \zeta^{(1)}$. If $\zeta \in \Sigma_S$, we must have

$$\begin{aligned}(\tau^1 + b\tau^2)^2 &= \mu|\xi^1 + b\xi^2|^2 \\ &\Rightarrow (\lambda + \mu)b^2 + 2((2\mu + \lambda) - \mu\xi^1 \cdot \xi^2))b + (\lambda + \mu) = 0.\end{aligned}$$

Because we assumed $\lambda + \mu > 0$, the equation above is quadratic and the determinant is positive if $\xi^1 \cdot \xi^2 \neq 1$, which is automatically true by the transversal intersection assumption. In this case, we get two real distinct roots b_+, b_- and two co-vectors in $\Sigma_S \cap N^*\mathscr{Z}_{PP}$

$$\zeta^+ = \zeta^{(1)} + b_+\zeta^{(2)}, \quad \zeta^- = \zeta^{(1)} + b_-\zeta^{(2)}.$$

Similarly, we have

Lemma 2 *Assume that $\mathscr{P}_1$ intersects $\mathscr{S}_2$ transversally at $\mathscr{Z}_{PS} \neq \emptyset$. Then*

$$\nabla \cdot \mathscr{G}_{PS} \in I^{\mu+2,\mu+2}(N^*\mathscr{Z}_{PS}, N^*\mathscr{P}_1) + I^{\mu+2,\mu+2}(N^*\mathscr{Z}_{PS}, N^*\mathscr{S}_2).$$

For any $p \in \mathscr{Z}_{PS}$, there exists a unique $\zeta_+ \in \Sigma_P \backslash N^*\mathscr{P}_1$ and $\zeta_- \in \Sigma_S \backslash N^*\mathscr{S}_2$ at p. The same conclusion holds for $\mathscr{G}_{SP}$.

Proof Let $(t, x) \in \mathscr{Z}_{PS}$ and $\zeta^{(1)} \in N^*\mathscr{P}_1, \zeta^{(2)} \in N^*\mathscr{S}_2$. We write $\zeta^{(i)} = (\tau^i, \xi^i), \xi^i \in \mathbb{R}^3, |\xi^i| = 1, i = 1, 2$. Then we have

$$(\tau^1)^2 = (\lambda + 2\mu)|\xi^1|^2, \quad (\tau^2)^2 = \mu|\xi^2|^2.$$

Now consider vectors $\zeta = \zeta^{(1)} + b\zeta^{(2)} \in N^*\mathscr{Z}_{PS}, b \in \mathbb{R}$. If $\zeta \in \Sigma_P$, we must have

$$\begin{aligned}(\tau^1 + b\tau^2)^2 &= (\lambda + 2\mu)|\xi^1 + b\xi^2|^2 \\ &\Rightarrow b^2(\lambda + \mu) + 2b(\xi^1 \cdot \xi^2(\lambda + 2\mu) - \sqrt{\mu(\lambda + 2\mu)}) = 0.\end{aligned}$$

The equation has two real solutions. One is $b = 0$ corresponding to the P vector $\zeta^{(1)}$ and $b_P \neq 0$ corresponding to a new vector in Σ_P. Now consider the vector ζ in Σ_S. We arrive at the equation

$$\begin{aligned}(\tau^1 + b\tau^2)^2 &= \mu|\xi^1 + b\xi^2|^2 \\ &\Rightarrow 2b(\sqrt{(\lambda + 2\mu)\mu} - \mu\xi^1 \cdot \xi^2) + (\lambda + \mu) = 0.\end{aligned}$$

So we get one non-trivial solution b_S. Thus, we conclude that $N^*\mathscr{Z}_{PS}$ has one P vector and one S vector. Similar conclusion holds for $\mathscr{G}^{SP}$.

Finally, we have

Lemma 3 *Assume that $\mathscr{S}_1$ intersects $\mathscr{S}_2$ transversally at $\mathscr{Z}_{SS} \neq \emptyset$. Then*

$$\nabla \cdot \mathscr{G}_{SS} \in I^{\mu+2,\mu+2}(N^*\mathscr{Z}_{SS}, N^*\mathscr{S}_1) + I^{\mu+2,\mu+2}(N^*\mathscr{Z}_{SS}, N^*\mathscr{S}_2), \text{ and}$$

1. $(N^*\mathscr{Z}_{SS} \backslash (N^*\mathscr{S}_1 \cup N^*\mathscr{S}_2)) \cap \Sigma_S = \emptyset$.
2. *For $p \in \mathscr{Z}_{SS}$, there are two linearly independent vectors $\zeta_+, \zeta_- \in N^*\mathscr{Z}_{SS} \cap \Sigma_P$ if the following interaction condition holds:*

$$\text{for } \zeta^i = (\tau^i, \xi^i) \in N_p^*\mathscr{S}_i, i = 1, 2, \text{ we have } \cos(\xi^1, \xi^2) < \frac{-\lambda}{\lambda + 2\mu}. \quad \text{(I)}$$

Proof Let $(t, x) \in \mathscr{Z}_{SS}$ and $\zeta^{(1)} \in N^*\mathscr{S}_1, \zeta^{(2)} \in N^*\mathscr{S}_2$. We write $\zeta^{(i)} = (\tau^i, \xi^i), \xi^i \in \mathbb{R}^3, i = 1, 2$ so that $|\xi^i| = 1$. We have $(\tau^i)^2 = \mu|\xi^i|^2 = \mu, \quad i = 1, 2.$ Now consider vectors $\zeta = \zeta^{(1)} + b\zeta^{(2)} \in N^*\mathscr{S}_{12}, b \in \mathbb{R}$. If $\zeta \in \Sigma_S$, we must have

$$\begin{aligned}(\tau^1 + b\tau^2)^2 &= (\lambda + 2\mu)|\xi^1 + b\xi^2|^2 \\ &\Rightarrow (\lambda + \mu)b^2 + 2((\lambda + 2\mu)\xi^1 \cdot \xi^2 - \mu)b + (\lambda + \mu) = 0. \quad (23)\end{aligned}$$

The equation has two distinct real roots $b_\pm$ if

$$\xi^1 \cdot \xi^2 < \frac{-\lambda}{\lambda + 2\mu}.$$

In this case, we get two P vectors in $N^*\mathscr{Z}_{SS}$.

We remark that by our assumption $\lambda+\mu > 0, \mu > 0$, we have $-\lambda/(\lambda+2\mu) \in (-1, 0)$. Thus one can find ζ^1, ζ^2 at $p \in \mathscr{Z}_{SS}$ so that the interaction condition holds.

Next, let's recall the microlocal parametrix for $Pu = f$ on $\mathbb{R} \times \mathbb{R}^3$. Let Diag $= \{(z, z') \in \mathbb{R}^4 \times \mathbb{R}^4 : z = z'\}$ be the diagonal of the product space and N^*Diag be the conormal bundle minus the zero section. We regard the symbols $\mathbf{p}_{P/S}(z, \zeta)$ as functions on the product space. Then we denote by Λ^P, Λ^S the flow out of N^*Diag under $H_{\mathbf{p}_P}, H_{\mathbf{p}_S}$. So $\Lambda^{P/S}$ are Lagrangian submanifolds of $T^*(\mathbb{R}^4 \times \mathbb{R}^4)$. We know that the system P is decomposed to the diagonal form. So according to [15], there exits a distribution

$$\begin{aligned}&Q_{sour} = Q^P_{sour} + Q^S_{sour},\\ &Q^P_{sour} \in I^{-\frac{3}{2},-\frac{1}{2}}(N^*\text{Diag}, \Lambda^P), \quad Q^S_{sour} \in I^{-\frac{3}{2},-\frac{1}{2}}(N^*\text{Diag}, \Lambda^S)\end{aligned}$$

such that $PQ_{sour} = \text{Id}$ up to a smoothing term. Here, the subscript $sour$ stands for the source problem. In the following, we denote

$$\begin{aligned}\Lambda^{PPS} &= \Lambda^S \circ N^*\mathscr{Z}_{PP}, \quad &\Lambda^{SSP} &= \Lambda^P \circ N^*\mathscr{Z}_{SS},\\ \Lambda^{PSP} &= \Lambda^P \circ N^*\mathscr{Z}_{PS}, \quad &\Lambda^{PSS} &= \Lambda^S \circ N^*\mathscr{Z}_{PS},\\ \Lambda^{SPP} &= \Lambda^P \circ N^*\mathscr{Z}_{SP}, \quad &\Lambda^{SPS} &= \Lambda^S \circ N^*\mathscr{Z}_{SP}.\end{aligned}$$

Again, because of the no conjugate point assumption, these are conormal bundles. In fact, $\Lambda^\bullet = N^*\mathscr{Z}_\bullet, \bullet = PPS, SSP, PSP, PSS, SPP, SPS$ where $\mathscr{Z}_\bullet$ are codimension one submanifolds of $\mathbb{R} \times \overline{\Omega}$.

Theorem 3 *Suppose $u^{(1)}, u^{(2)}$ are distorted plane waves in Definition 1.*

1. *The solution to (8) can be decomposed as*

$$u^{(12)} = u^{PPS} + u^{PSP} + u^{PSS} + u^{SPP} + u^{SPS} + u^{SSP},$$

such that microlocally away from $\Lambda_1^P \cup \Lambda_2^P \cup \Lambda_1^S \cup \Lambda_2^S$*, we have*

$$u^\bullet \in I^{2\mu+\frac{5}{2}}(\Lambda^\bullet), \quad \bullet = PPS, PSP, PSS, SPP, SPS, SSP.$$

Moreover, u^{SSP} *is smooth on* Λ^{SSP} *unless* $\mathscr{S}_1, \mathscr{S}_2$ *satisfy the interaction condition.*

2. *If* $\mathscr{Z}_\bullet$ *intersect* $\mathbb{R} \times \partial\Omega$ *transversally at* $\mathscr{Y}_\bullet$*, then*

$$\partial_{\epsilon_1}\partial_{\epsilon_2}\Lambda(f_\epsilon) \in I^{2\mu+\frac{9}{4}}(N_\partial^*\mathscr{Y}_\bullet)$$

are conormal distributions.

3. *Consider the symbol at* $\mathscr{Y}_\bullet = \mathscr{Y}_{PPS}$*. Let* $(z_0, \zeta_0) \in T^*Z$ *and the bicharacteristic from* (z_0, ζ_0) *intersect* T^*Y *transversally. Let* $(z, \zeta) = \Lambda^P(z_0, \zeta_0)$ *and* $(z_|, \zeta_|) = R_0(z, \zeta)$ *with* R_0 *the canonical relation of the restriction operator. Then the principal symbol satisfies*

$$\sigma(\partial_{\epsilon_1}\partial_{\epsilon_2}\Lambda(f_\epsilon))(z_|, \zeta_|) = \sigma(\rho_0)(z_|, \zeta_|, z, \zeta)Q^S_{sour}(z, \zeta, z_0, \zeta_0)\sigma(\nabla \cdot \mathscr{G}^{PP})(z_0, \zeta_0),$$

where Q^S_{sour} *and* $\sigma(\rho_0)$ *are* 3×3 *invertible matrices. Similar statements hold for* $\mathscr{Y}_\bullet, \bullet = PSP, PSS, SPP, SPS, SSP$.

Proof We analyze u^{PPS} and the others are similar. We know that away from Λ_1^P, Λ_2^P, $\nabla \cdot \mathscr{G}^{PP} \in I^{2\mu-2}(N^*\mathscr{Z}_{PP})$. Because $N^*\mathscr{Z}_{PP}$ intersect Σ_S transversally, we can apply Proposition 2.2 of [5] to get

$$u^{PPS} = Q^S(\nabla \cdot \mathscr{G}^{PP}) \in I^{\mu+2+\mu+2-\frac{3}{2},-\frac{1}{2}}(N^*\mathscr{Z}_{PP}, \Lambda^{PPS})$$

modulo a distribution whose wave front set is contained in a neighborhood of Λ_1^P, Λ_2^P. Thus, away from $\mathscr{Z}_{PP}$, $u^{PPS} \in I^{2\mu+\frac{5}{2}}(\Lambda^{PPS})$.

Next, if Λ^{PPS} intersect the boundary Y transversally, we see that $\partial_{\epsilon_1}\partial_{\epsilon_2}\Lambda(f_\epsilon) = \rho_0(u^{PPS})$ near the intersection. By the composition of FIOs, we know the term is a conormal distribution with order $-1/4$ less than that of u^{PPS}.

We remark that because $\mathscr{Z}_\bullet$ are of codimension one, the singularities of the nonlinear response $u^\bullet$ above are of the same type as a distorted plane wave, see Fig. 2. Also, if $\partial\Omega$ is strictly convex with respect to $g_{P/S}$, the intersection of $\mathscr{Z}_\bullet$ and $\mathbb{R} \times \partial\Omega$ is transversal. We also remark that u^{PPS} can be regarded as consisting of two waves in view of Lemma 1. The same is true for u^{SSP} in view of Lemma 3.

4.3 Symbols of the nonlinear responses

We determine the symbol of the interaction terms and show that they are not always vanishing. This would confirm the generation of new waves. Roughly, there are three kinds of interactions so we split the section to three subsections.

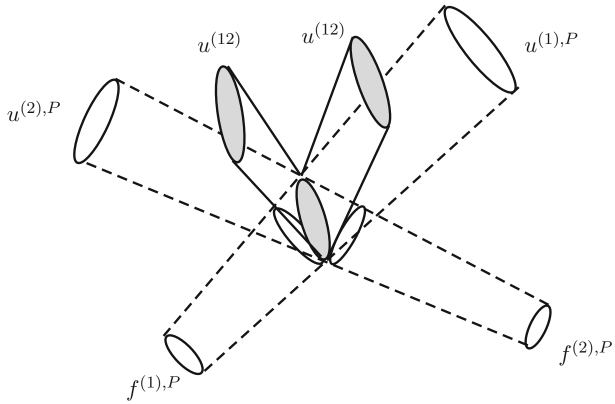
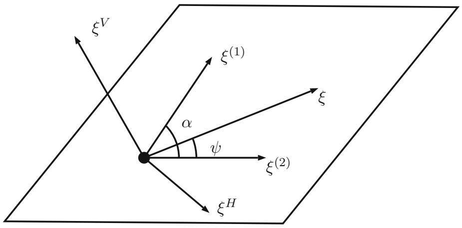


Fig. 2 Illustration of the interaction of two P waves. The picture is in $\mathbb{R}^3$. The white ellipses show the evolution of the singular supports of two P waves for different time $t > 0$ along the two paths. The gray ellipses show the generation and evolution of the wave fronts of the generated S wave

Fig. 3 The interaction plane for P–P wave interactions



4.3.1 P–P interactions

We take $u^{(\bullet),P} \in I^{\mu}(\Lambda_{\bullet}^{P})$, $\bullet = 1, 2$, and consider the singularities of $\mathscr{G}^{PP}$. For ease of calculation, we introduce some quantities for the interaction. Let $z \in \mathscr{P}_1 \cap \mathscr{P}_2$ and $(z, \zeta^1) \in \Lambda_1^P$, $(z, \zeta^2) \in \Lambda_2^P$. Assume that $\zeta = \zeta^1 + \zeta^2 \in \Sigma_S$. Let $\zeta^{\bullet} = (\tau^{\bullet}, \xi^{\bullet})$, $\xi^{\bullet} \in \mathbb{R}^3$, $\bullet = 1, 2$. We call the plane determined by ξ^1, ξ^2 the *interaction plane*. Then $\xi = \xi^1 + \xi^2$. Because we consider the S wave, we let ξ^H be a unit vector in the interaction plane perpendicular to ξ and ξ^V be a unit vector orthogonal to the interaction plane. We define the angles α, ψ through

$$\xi^1 \cdot \xi^2 = |\xi^1||\xi^2| \cos\alpha, \quad \xi \cdot \xi^2 = |\xi||\xi^2| \cos\psi.$$

See Fig. 3. The angles α, ψ and the relations have been used in the literatures (e.g. [11]) and they are physically useful.

We consider the term $\mathscr{G}(u^{(1),P}, u^{(2),P})$. The symbol of $u^{(\bullet),P}$ at $(z, \zeta^{\bullet}) \in \Lambda_{\bullet}^{P}$ is the projection of $\sigma(u^{(\bullet)})(z, \zeta^{\bullet})$ by $\sigma(\Pi_P)$ at z along the $\xi^{\bullet}$, $\bullet = 1, 2$ direction. Thus we can write $\sigma(u^{(\bullet),P})(z, \zeta^{\bullet}) = a^{\bullet}\xi^{\bullet}$ for some constant $a^{\bullet}$. Then consider

$$\widetilde{e}_{mn}^{\bullet} = \frac{1}{2}\left(\frac{\partial u_m^{(\bullet),P}}{\partial x_n} + \frac{\partial u_n^{(\bullet),P}}{\partial x_m}\right) \in I^{\mu+1}(\Lambda_{\bullet}^{P}), \quad \bullet = 1, 2.$$

The corresponding symbol of $\widetilde{e}^{\bullet}$ is

$$U^{\bullet} = \imath a_{\bullet} \xi^{\bullet,T} \xi^{\bullet}, \quad \bullet = 1, 2.$$

Here, $\imath^2 = -1$ and $\xi^{\bullet}$ are regarded as row vectors hence $\xi^{\bullet,T}\xi^{\bullet}$ are 3×3 symmetric matrices. Let $\zeta = \zeta^1 + \zeta^2$. We denote the principal symbol of $\mathscr{G}(u^{(1),P}, u^{(2),P}) \in I^{\mu+1}(N^*\mathscr{Z}_{PP})$ at $(z, \zeta) \in N^*\mathscr{Z}_{PP} \cap \Sigma_S$ by $\mathbf{g}(z, \zeta)$. We recall the symbol calculation from [13, Lemma 3.3]. For $u^{(\bullet),P} \in I^{\mu}(N^*\mathscr{P}_{\bullet})$, consider the principal symbol of $u^{(1),P}u^{(2),P} \in I^{2\mu+3}(N^*\mathscr{Z}_{PP}\backslash N^*\mathscr{P}_1 \cup N^*\mathscr{P}_2)$ in local coordinates of $p \in \mathscr{Z}_{PP}$. For $\zeta = \zeta_1 + \zeta_2 \in N^*\mathscr{Z}_{PP}$ with $\zeta_{\bullet} \in N^*\mathscr{P}_{\bullet}$, we have

$$\sigma(u^{(1),P}u^{(2),P})(z, \zeta) = \sigma(u^{(1),P})(z, \zeta_1)\sigma(u^{(2),P})(z, \zeta_2).$$

Here, we absorbed the $(2\pi)^{-1}$ factor in [13, Lemma 3.3] to the symbols. Then we use [22, Lemma 4.1] and the expression (24) of $G(u, u)$ to get

$$\begin{aligned} -\mathbf{g}_{mn} &= \lambda U^1_{jj}U^2_{mn} + \lambda U^2_{jj}U^1_{mn} + \lambda U^1_{kj}U^2_{kj}\delta_{mn} \\ &\quad + 2\mu U^1_{nj}U^2_{mj} + \mu U^1_{km}U^2_{kn} + 2\mu U^2_{nj}U^1_{mj} + \mu U^2_{km}U^1_{kn} \\ &\quad + A[U^1_{mj}U^2_{nj} + U^2_{mj}U^1_{nj}] + B(2U^1_{jj}U^2_{mn} + 2U^2_{jj}U^1_{mn} \\ &\quad + 2U^1_{ij}U^2_{ij}\delta_{mn}) + C2U^1_{ii}U^2_{jj}\delta_{mn} \\ &= (\lambda + 2B)a_1a_2[|\xi^1|^2\xi^2_m\xi^2_n + |\xi^2|^2\xi^1_m\xi^1_n + (\xi^1 \cdot \xi^2)^2\delta_{mn}] \\ &\quad + (A + 3\mu)a_1a_2[\xi^1_m\xi^1_k\xi^2_k\xi^2_n + \xi^2_m\xi^2_k\xi^1_k\xi^1_n] + 2Ca_1a_2|\xi^1|^2|\xi^2|^2\delta_{mn}. \end{aligned} \tag{24}$$

(The negative sign is due to the symbol of two derivatives.) Then we get

$$\mathbf{h}(z, \zeta) = \sigma(\nabla \cdot \mathscr{G}(u^{(1),P}, u^{(2),P}))(z, \zeta) = \imath \mathbf{g}(z, \zeta)\xi.$$

Because we consider the S wave propagation, we project the symbol $\mathbf{h}$ along the ξ^H and ξ^V directions, which are denoted by $\mathbf{h}^{SH}$, $\mathbf{h}^{SV}$ respectively. Then the symbol $\mathbf{h}^{S\bullet}$ are

$$|\xi^{\bullet}|^{-2}(\xi^{\bullet}\mathbf{g}(z, \zeta)\xi)\xi^{\bullet} = (\xi^{\bullet}_m \mathbf{g}_{mn}\xi_n)\xi^{\bullet}, \quad \bullet = V, H. \tag{25}$$

We first compute the symbol $\mathbf{h}^{SV}$:

$$\begin{aligned} \imath \mathbf{h}^{SV}(z, \zeta) &= (\lambda + 2B)a_1a_2[|\xi^1|^2(\xi^V \cdot \xi^2)(\xi^2 \cdot \xi) + |\xi^2|^2(\xi^V \cdot \xi^1)(\xi^1 \cdot \xi)]\xi^V \\ &\quad + (3\mu + A)a_1a_2[(\xi^V \cdot \xi^1)(\xi^1 \cdot \xi^2)(\xi^2 \cdot \xi) \\ &\quad + (\xi^V \cdot \xi^2)(\xi^2 \cdot \xi^2)(\xi^1 \cdot \xi)]\xi^V = 0, \end{aligned}$$

because of ξ^V is perpendicular to the interaction plane. Next we consider the symbol $\mathbf{h}^{SH}$:

$$\begin{aligned}\imath \mathbf{h}^{SH}(z,\zeta) &= (\lambda+2B)a_1a_2[|\xi^1|^2((\xi^H\cdot\xi^2)(\xi^2\cdot\xi)+|\xi^2|^2(\xi^H\cdot\xi^1)(\xi^1\cdot\xi)\\ &\quad+(\xi^1\cdot\xi^2)^2(\xi^H\cdot\xi)]\xi^H\\ &\quad+(3\mu+A)a_1a_2[(\xi^H\cdot\xi^1)(\xi^1\cdot\xi^2)(\xi^2\cdot\xi)+(\xi^H\cdot\xi^2)(\xi^2\cdot\xi^1)(\xi^1\cdot\xi)]\xi^H\\ &\quad+2Ca_1a_2|\xi^1|^2|\xi^2|^2(\xi^H\cdot\xi)\xi^H\\ &= a_1a_2|\xi^1|^2|\xi^2|^2|\xi|[(\lambda+2B)(\sin\psi\cos\psi-\sin(\alpha-\psi)\cos(\alpha-\psi))\\ &\quad+(A+3\mu)(-\sin(\alpha-\psi)\cos\alpha\cos\psi+\sin\psi\cos\alpha\cos(\alpha-\psi))]\xi^H.\end{aligned}$$

Using trigonometry identities, we obtain that

$$\mathbf{h}^{SH}(z,\zeta) = -\imath a_1a_2|\xi^1|^2|\xi^2|^2|\xi|(\lambda+3\mu+A+2B)\cos\alpha\sin(2\psi-\alpha)\xi^H.$$

If $\lambda+3\mu+A+2B\neq 0$, this is non-vanishing for (ψ,α) in any open set of $(0,\pi)^2$. In this sense, we call the symbol *generically non-vanishing*. We also observe that in the principal symbol of $\mathscr{G}^{PP}$ the information of $C(x)$ is lost.

4.3.2 P–S interactions

Consider the term $\mathscr{G}^{PS}=\mathscr{G}(u^{(1),P},u^{(2),S})$. The analysis for $\mathscr{G}^{SP}$ is the same. For simplicity, we let $u^{(1)}=u^{(1),P}$ and $u^{(2)}=u^{(2),S}$. For the principal symbol of $u^{(2),S}$ at $(z,\zeta)=(t,x;\tau,\xi)\in\Lambda^S$, we observe that

$$\sigma(\widetilde{e}_{ii}^{(2)})(z,\zeta)=\sum_{i=1}^{3}\xi_i\frac{|\xi^2|\delta_{il}-\xi_i\xi_l}{|\xi|^2}\imath\sigma(u_l^{(2),S})=0.$$

(Another way to see this is that the S component of u is divergence free.) This type of term appears in $C\widetilde{e}_{ii}\widetilde{e}_{jj}\delta_{mn}$ of $G(u,u)$ so $C(x)$ does not appear in the symbols for interactions involving S waves. Therefore, before we compute the symbols explicitly, we proved

Proposition 3 *For the two distorted plane waves $u^{(1)},u^{(2)}$ in Definition 1, the principal symbols of the corresponding terms $\mathscr{G}^{\bullet,\bullet}=PP,PS,SP,SS$ are independent of $C(x)$. So are the symbols of the nonlinear responses $u^{\bullet,\bullet}=PPS,PSP,PSS,SPS,SPP,SSP$.*

Now we proceed to determine the principal symbol of $\mathscr{G}^{PS}$. We again introduce the interaction plane to simplify the calculation, see Fig. 4. Let $z\in\mathscr{P}_1\cap\mathscr{S}_2$ and $(z,\zeta^1)\in\Lambda_1^P$, $(z,\zeta^2)\in\Lambda_2^S$. Let $\zeta^i=(\tau^i,\xi^i)$, $i=1,2$ as before. We call the plane determined by ξ^1,ξ^2 the interaction plane. Let ξ^H be a unit vector in the interaction plane orthogonal to ξ^2. Then let ξ^V be a unit vector orthogonal to the interaction plane.

We first consider the P mode of $\mathscr{G}^{PS}$. Assume that $\zeta^P=\zeta^1+\zeta^2\in\Sigma_P$ and let $\zeta=(\tau^P,\xi^P)$. We define the angles α,ψ through

$$\xi^1\cdot\xi^2=|\xi^1||\xi^2|\cos\alpha,\quad \xi\cdot\xi^P=|\xi||\xi^P|\cos\psi,$$

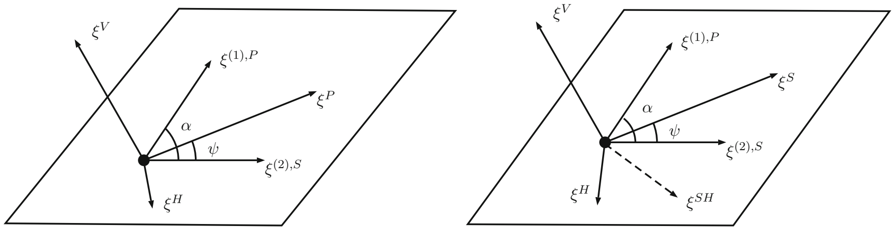


Fig. 4 The interaction plane for P–S wave interactions. Left: picture for the P mode. Right: picture for the S mode

see the left figure of Fig. 4. Now we can express the principal symbols of $u^{(1)}, u^{(2)}$ in terms of these quantities. We let $\sigma(u^{(1),P})(z,\zeta^1) = a\xi^1$ for some constant a and we decompose $\sigma(u^{(2),S})(z,\zeta^2) = b_H\xi^H + b_V\xi^V$ for some constants $b_\bullet$. We let $U^1 = a\xi^{1,T}\xi^1$ so that the principal symbol of $\widetilde{e}^1$ is $\imath U^1$. Next we let

$$W_{mn} = \xi_n^2(b_H\xi_m^H + b_V\xi_m^V) = \xi_n^2 b_H\xi_m^H + \xi_n^2 b_V\xi_m^V = W_{mn}^H + W_{mn}^V$$

corresponding to the H, V decomposition. We observe that the principal symbol $\sigma(\frac{\partial u_m^{(2),S}}{\partial x_n})$ is $\imath W_{mn}$. Now we define

$$\begin{aligned} U^2 &= \frac{1}{2}(W + W^T) = \frac{1}{2}\xi^{2,T}(b_H\xi^H + b_V\xi^V) + \frac{1}{2}(b_H\xi^{H,T} + b_V\xi^{V,T})\xi^2 \\ &= \frac{1}{2}b_H(\xi^{2,T}\xi^H + \xi^{H,T}\xi^2) + \frac{1}{2}b_V(\xi^{2,T}\xi^V + \xi^{V,T}\xi^2) = U^H + U^V, \end{aligned}$$

where U^H, U^V are defined by the second line. So the principal symbol of $\widetilde{e}^{(2)}$ is $\imath U^2$. We remark that the U matrices are symmetric but W matrices are not. Because of the H, V decomposition, we will write $\sigma(\mathscr{G}^{PS})(z,\zeta) = \mathbf{g}^H + \mathbf{g}^V$ where $\mathbf{g}^\bullet, \bullet = H, V$ are defined as

$$\begin{aligned} -\mathbf{g}_{mn}^\bullet &= \lambda U_{jj}^1 W_{mn}^\bullet + \lambda U_{kj}^1 W_{kj}^\bullet\delta_{mn} + 2\mu U_{nj}^1 W_{mj}^\bullet + \mu U_{km}^1 W_{kn}^\bullet + 2\mu U_{nj}^\bullet U_{mj}^1 \\ &\quad + \mu W_{km}^\bullet U_{kn}^1 + A[U_{mj}^1 U_{nj}^\bullet + U_{mj}^\bullet U_{nj}^1] + B(2U_{jj}^1 U_{mn}^\bullet + 2U_{ij}^1 U_{ij}^\bullet\delta_{mn}) \\ &= \lambda ab_\bullet(|\xi_1|^2\xi_m^\bullet\xi_n^2 + \xi_k^1\xi_j^1\xi_k^\bullet\xi_j^2\delta_{mn}) \\ &\quad + 2\mu ab_\bullet\xi_n^1\xi_j^1\xi_m^\bullet\xi_j^2 + \mu ab_\bullet\xi_k^1\xi_m^1\xi_k^\bullet\xi_n^2 + \mu ab_\bullet(\xi_n^\bullet\xi_j^2 + \xi_j^\bullet\xi_n^2)\xi_m^1\xi_j^1 \\ &\quad + \mu ab_\bullet\xi_k^\bullet\xi_m^2\xi_k^1\xi_n^1 + Aab_\bullet\frac{1}{2}[\xi_m^1\xi_j^1(\xi_n^\bullet\xi_j^2 + \xi_j^\bullet\xi_n^2) + (\xi_m^\bullet\xi_j^2 + \xi_j^\bullet\xi_m^2)\xi_n^1\xi_j^1] \\ &\quad + Bab_\bullet(|\xi_1|^2(\xi_m^\bullet\xi_n^2 + \xi_n^\bullet\xi_m^2) + \xi_i^1\xi_j^1(\xi_i^\bullet\xi_j^2 + \xi_j^\bullet\xi_i^2)\delta_{mn}). \end{aligned} \tag{26}$$

Then we get

$$\mathbf{h}(z,\zeta) = \sigma(\nabla\cdot\mathscr{G}(u^{1,P}, u^{2,S}))(z,\zeta) = \imath(\mathbf{g}^H(z,\zeta) + \mathbf{g}^V(z,\zeta))\xi^P.$$

Finally, we project the symbol to ξ^P direction to get the symbol of the P mode:

$$\mathbf{h}^{\bullet P} = |\xi^P|^{-2}(\xi^P \mathbf{g}^{\bullet}(z,\zeta)\xi^P)\xi^P, \quad \bullet = H, V.$$

We compute

$$\begin{aligned}
&\imath \mathbf{h}^{HP}(z,\zeta)\\
&= |\xi^P|^{-2} a b_H (\lambda + 2B)[|\xi^1|^2(\xi^P\cdot\xi^2)(\xi^P\cdot\xi^H) + (\xi^1\cdot\xi^2)(\xi^1\cdot\xi^H)|\xi^P|^2]\xi^P\\
&\quad + |\xi^P|^{-2} a b_H (3\mu + A)[(\xi^P\cdot\xi^2)(\xi^1\cdot\xi^P)(\xi^1\cdot\xi^H) + (\xi^P\cdot\xi^H)(\xi^1\cdot\xi^P)(\xi^1\cdot\xi^2)]\xi^P\\
&= a b_H |\xi^1|^2|\xi^2||\xi^H|[(\lambda + 2B)(-\cos\psi\sin\psi - \cos\alpha\sin\alpha)\\
&\quad + (A + 3\mu)(-\cos\psi\cos(\alpha-\psi)\sin\alpha - \sin\psi\cos(\alpha-\psi)\cos\alpha)]\xi^P.
\end{aligned}$$

We observe that

$$\begin{aligned}
&-\cos\psi\cos(\alpha-\psi)\sin\alpha - \sin\psi\cos(\alpha-\psi)\cos\alpha = -\cos(\alpha-\psi)\sin(\alpha+\psi)\\
&= -\cos\psi\sin\psi - \cos\alpha\sin\alpha.
\end{aligned}$$

Thus we have

$$\mathbf{h}^{HP}(z,\zeta) = \imath a b_H |\xi^1|^2|\xi^2|(\lambda + 2B + A + 3\mu)\cos(\alpha-\psi)\sin(\alpha+\psi)\xi^P.$$

This term is generically non-vanishing when $\lambda + 2B + A + 3\mu \neq 0$.

Next, consider the interactions with the V components of $u^{(2),S}$.

$$\begin{aligned}
&\imath \mathbf{h}^{VP}(z,\zeta)\\
&= |\xi^P|^{-2} a b_V (\lambda + 2B)[|\xi^1|^2(\xi^P\cdot\xi^2)(\xi^P\cdot\xi^V) + (\xi^1\cdot\xi^2)(\xi^1\cdot\xi^V)|\xi^P|^2]\xi^P\\
&\quad + |\xi^P|^{-2} a b_V (3\mu + A)[(\xi^P\cdot\xi^2)(\xi^1\cdot\xi^P)(\xi^1\cdot\xi^V)\\
&\quad + (\xi^P\cdot\xi^V)(\xi^1\cdot\xi^P)(\xi^1\cdot\xi^2)]\xi^P = 0.
\end{aligned}$$

Thus we conclude that the symbol of u^{PSP} at (z,ζ) is given by $\mathbf{h}^{HP}$ and the term is generically non-vanishing.

It remains to consider the generation of S mode from the P–S interaction. In this case, we let $\xi = \xi^S$ and ξ^{SH} be the unit vector in the interaction plane orthogonal to ξ^S. We decompose u to the plane determined by ξ^{SH} and ξ^V, see the right picture of Fig. 4. The computation of $\mathbf{g}$ is the same as (27) and we have the symbol of $\mathscr{G}^{PS}$

$$\mathbf{h}(z,\zeta) = \sigma(\nabla\cdot\mathscr{G}(u^{(1),P}, u^{(2),S}))(z,\zeta) = \imath(\mathbf{g}^H(z,\zeta) + \mathbf{g}^V(z,\zeta))\xi^S.$$

We project the symbol to ξ^*, $* = V, SH$ directions to get the symbol of the S mode:

$$\mathbf{h}^{\bullet *} = |\xi^*|^{-2}\imath(\xi^* \mathbf{g}^{\bullet}(z,\zeta)\xi^S)\xi^*, \quad \bullet = V, H, \quad * = V, SH.$$

We compute

$$
\begin{aligned}
&\imath \mathbf{h}^{\bullet *}(z,\zeta)\\
&= ab_{\bullet}\lambda[|\xi^1|^2(\xi^*\cdot\xi^{\bullet})(\xi^2\cdot\xi^S)+(\xi^1\cdot\xi^{\bullet})(\xi^1\cdot\xi^2)(\xi^*\cdot\xi^S)]\xi^*\\
&\quad+2\mu ab_{\bullet}(\xi^*\cdot\xi^{\bullet})(\xi^1\cdot\xi^2)(\xi^1\cdot\xi^S)\xi^*+\mu ab_{\bullet}(\xi^1\cdot\xi^*)(\xi^1\cdot\xi^{\bullet})(\xi^2\cdot\xi^S)\xi^*\\
&\quad+\mu ab_{\bullet}[(\xi^1\cdot\xi^*)(\xi^1\cdot\xi^2)(\xi^{\bullet}\cdot\xi^S)+(\xi^1\cdot\xi^*)(\xi^1\cdot\xi^{\bullet})(\xi^2\cdot\xi^S)\\
&\quad+(\xi^2\cdot\xi^*)(\xi^1\cdot\xi^{\bullet})(\xi^1\cdot\xi^S)]\xi^*\\
&\quad+Aab_{\bullet}\frac{1}{2}[(\xi^1\cdot\xi^*)(\xi^1\cdot\xi^2)(\xi^{\bullet}\cdot\xi^S)+(\xi^1\cdot\xi^*)(\xi^1\cdot\xi^{\bullet})(\xi^2\cdot\xi^S)\\
&\quad+(\xi^{\bullet}\cdot\xi^*)(\xi^1\cdot\xi^2)(\xi^1\cdot\xi^S)\\
&\quad+(\xi^2\cdot\xi^*)(\xi^{\bullet}\cdot\xi^1)(\xi^1\cdot\xi^S)]\xi^*\\
&\quad+Bab_{\bullet}[|\xi^1|^2(\xi^*\cdot\xi^{\bullet})(\xi^2\cdot\xi^S)+|\xi^1|^2(\xi^{\bullet}\cdot\xi^S)(\xi^2\cdot\xi^*)\\
&\quad+2(\xi^1\cdot\xi^{\bullet})(\xi^1\cdot\xi^2)(\xi^*\cdot\xi^S)]\xi^*.
\end{aligned}
$$

We observe that if $\bullet = V, * = SH$ or $\bullet = H, * = V$ then the term must be zero. So it suffices to consider two cases. After some calculations, we find that

$$
\begin{aligned}
\imath \mathbf{h}^{HSH}(z,\zeta) &= ab_H|\xi^1|^2|\xi^2||\xi^S|[\lambda\cos^2\psi+\mu(\cos(2\psi)+\cos^2\psi)\\
&\quad+\frac{1}{2}A\cos(2\psi)+B(\cos^2\psi-\sin^2\psi)]\xi^{SH}\\
&= ab_H|\xi^1|^2|\xi^2||\xi^S|\left[\left(\lambda+2\mu+B+\frac{1}{2}A\right)\cos^2\psi-\left(\mu+B+\frac{1}{2}A\right)\sin^2\psi\right]\xi^{SH}.
\end{aligned}
$$

The following lemma is straightforward.

Lemma 4 *For $\psi \in (0,\pi)$, consider the vector $\mathbf{v}(\psi) = [\cos^2\psi, \sin^2\psi]$. Then $\det(\mathbf{v}(\psi_1), \mathbf{v}(\psi_2))$ is non-vanishing for ψ_1, ψ_2 in any open subset of $(0,\pi)^2$.*

In this sense, we say that the symbol $\mathbf{h}^{HSH}$ is generically nonvanishing if $\lambda + 2\mu + \frac{1}{2}A + B \neq 0$ or $\mu + \frac{1}{2}A + B \neq 0$.

Next, we calculate that

$$
\begin{aligned}
\imath \mathbf{h}^{VV}(z,\zeta) &= ab_V|\xi^1|^2|\xi^2||\xi^S|[\lambda\cos\psi+2\mu\cos\alpha\cos(\alpha-\psi)\\
&\quad+\frac{1}{2}A\cos\alpha\cos(\alpha-\psi)+B\cos\psi]\xi^V\\
&= ab_V|\xi^1|^2|\xi^2||\xi^S|\left[(\lambda+B)\cos\psi+\left(2\mu+\frac{1}{2}A\right)\cos\alpha\cos(\alpha-\psi)\right]\xi^V.
\end{aligned}
$$

Similarly, we conclude that the term is generically non-vanishing if $\lambda + B \neq 0$ or $2\mu + \frac{1}{2}A \neq 0$.

4.3.3 S–S interactions

We let $u^{(1)} = u^{(1),S}$ and $u^{(2)} = u^{(2),S}$ and we consider $\mathscr{G}^{SS}$. We decompose the S modes according to the interaction plane, see Fig. 5. Let $z \in \mathscr{S}_1 \cap \mathscr{S}_2$ and $(z, \zeta^1) \in$

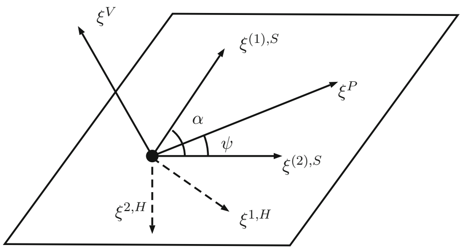


Fig. 5 The interaction plane for S–S wave interactions

Λ_1^S, $(z,\zeta^2)\in\Lambda_2^S$. Let $\zeta^i=(\tau^i,\xi^i)$, $i=1,2$ as before. We call the plane determined by ξ^1,ξ^2 the interaction plane. Let $\xi^{i,H}$ be a unit vector in the interaction plane orthogonal to ξ^i, $i=1,2$. Then let ξ^V be a unit vector orthogonal to the interaction plane. We can decompose $u^{(1)}, u^{(2)}$ to H, V modes.

We decompose $\sigma(u^{(i),S})(z,\zeta^2)=b_H^i\xi^{i,H}+b_V^i\xi^{i,V}$ for some $b_\bullet^i$ constants, $i=1,2$, $\bullet=H,V$. Similar to the previous case, we let

$$W_{mn}^i=\xi_n^i\left(b_H^i\xi_m^{i,H}+b_V^i\xi_m^{i,V}\right)=\xi_n^ib_H^i\xi_m^{i,H}+\xi_n^ib_V^i\xi_m^{i,V}=W_{mn}^{i,H}+W_{mn}^{i,V},\quad i=1,2,$$

corresponding to the H, V decomposition. The principal symbol $\sigma\left(\dfrac{\partial u_m^{(i),S}}{\partial x_n}\right)$ is $\imath W_{mn}^i$. Now we define

$$\begin{aligned}U^i&=\frac{1}{2}(W^i+W^{i,T})=\frac{1}{2}\xi^{i,T}\left(b_H^i\xi^{i,H}+b_V^i\xi^{i,V}\right)+\frac{1}{2}\left(b_H^i\xi^{i,H,T}+b_V^i\xi^{i,V,T}\right)\xi^i\\&=\frac{1}{2}b_H^i\left(\xi^{i,T}\xi^{i,H}+\xi^{i,H,T}\xi^i\right)+\frac{1}{2}b_V^i\left(\xi^{i,T}\xi^{i,V}+\xi^{i,V,T}\xi^i\right)=U^{i,H}+U^{i,V},\end{aligned}$$

So the principal symbol of $\widetilde{e}^{(i)}$ is $\imath U^i$. Because of the H, V decomposition, we will write

$$\sigma(\mathscr{G}^{PS})(z,\zeta)=\mathbf{g}^{HH}+\mathbf{g}^{HV}+\mathbf{g}^{VH}+\mathbf{g}^{VV},$$

where $\mathbf{g}^{*\bullet}$, $*,\bullet=$ H, V are defined as

$$\begin{aligned}-\mathbf{g}_{mn}^{*\bullet}&=\lambda W_{kj}^{1,*}W_{kj}^{2,\bullet}\delta_{mn}+2\mu U_{nj}^{1,*}W_{mj}^{2,\bullet}+\mu W_{km}^{1,*}W_{kn}^{2,\bullet}+2\mu U_{nj}^{2,\bullet}W_{mj}^{1,*}+\mu W_{km}^{2,\bullet}W_{kn}^{1,*}\\&\quad+A[U_{mj}^{1,*}U_{nj}^{2,\bullet}+U_{mj}^{2,\bullet}U_{nj}^{1,*}]+2BU_{ij}^{1,*}U_{ij}^{2,\bullet}\delta_{mn},\\&=\lambda\xi_k^{1,*}\xi_j^1\xi_k^{2,\bullet}\xi_j^2\delta_{mn}+\mu(\xi_n^{1,*}\xi_j^1+\xi_n^1\xi_j^{1,*})\xi_m^{2,\bullet}\xi_j^2+\mu\xi_k^{1,*}\xi_m^1\xi_k^{2,\bullet}\xi_n^2\\&\quad+\mu(\xi_n^{2,\bullet}\xi_j^2+\xi_n^2\xi_j^{2,\bullet})\xi_m^{1,*}\xi_j^1+\mu\xi_k^{2,\bullet}\xi_m^2\xi_k^{1,*}\xi_n^1\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{4}A[(\xi_m^{1,*}\xi_j^1+\xi_m^1\xi_j^{1,*})(\xi_n^{2,\bullet}\xi_j^2+\xi_n^2\xi_j^{2,\bullet})+(\xi_m^{2,\bullet}\xi_j^2+\xi_m^2\xi_j^{2,\bullet})(\xi_n^{1,*}\xi_j^1+\xi_n^1\xi_j^{1,*})]\\
&+\frac{1}{2}B(\xi_i^{1,*}\xi_j^1+\xi_i^1\xi_j^{1,*})(\xi_i^{2,\bullet}\xi_j^2+\xi_i^2\xi_j^{2,\bullet})\delta_{mn}.
\end{aligned}
\tag{27}
$$

These terms represents the interaction of all possible combinations of the H, V modes.

Remember that we are computing the P mode of $\mathscr{G}^{SS}$ when the interaction condition is satisfied. So we let $\zeta^P=\zeta^1+\zeta^2\in\Sigma^P$ and $\zeta^P=(\tau^P,\xi^P)$. As before, we get

$$
\begin{aligned}
\mathbf{h}(z,\zeta)&=\sigma(\nabla\cdot\mathscr{G}(u^{1,S},u^{2,S}))(z,\zeta)\\
&=\imath(\mathbf{g}^{HH}(z,\zeta)+\mathbf{g}^{HV}(z,\zeta)+\mathbf{g}^{VH}(z,\zeta)+\mathbf{g}^{VV}(z,\zeta))\xi^P.
\end{aligned}
$$

Finally, we project the symbol to ξ^P direction to get the symbol of the P mode:

$$
\mathbf{h}^\bullet=|\xi^P|^{-2}(\xi^P\imath\mathbf{g}^\bullet(z,\zeta)\xi^P)\xi^P,\quad \bullet=HH,HV,VH,VV.
$$

Because of the orthogonality, one can check that $\mathbf{h}^{HV}=\mathbf{h}^{VH}=0$ (details are omitted). We compute $\mathbf{h}^{HH}$, $\mathbf{h}^{VV}$ carefully. We have

$$
\begin{aligned}
\imath\mathbf{h}^{VV}&=b_V^1b_V^2|\xi^P|^{-2}[\lambda|\xi^P|^2(\xi^1\cdot\xi^2)(\xi^V\cdot\xi^V)+2\mu(\xi^1\cdot\xi^P)(\xi^V\cdot\xi^V)(\xi^2\cdot\xi^P)\\
&\quad+\frac{1}{2}A(\xi^1\cdot\xi^P)(\xi^V\cdot\xi^V)(\xi^2\cdot\xi^P)+B|\xi^P|^2(\xi^V\cdot\xi^V)(\xi^1\cdot\xi^2)]\xi^P\\
&=b_V^1b_V^2|\xi^1||\xi^2|\left[(\lambda+B)\cos\alpha+\left(\frac{1}{2}A+2\mu\right)\cos\psi\cos(\alpha-\psi)\right]\xi^P.
\end{aligned}
$$

This term is generically non-vanishing if $\lambda+B\neq 0$ or $\frac{1}{2}A+2\mu\neq 0$. At last, we compute

$$
\begin{aligned}
&\imath|\xi^P|^2\mathbf{h}^{HH}\\
&=\lambda b_H^1b_H^2(\xi^{1,H}\cdot\xi^{2,H})(\xi^1\cdot\xi^2)(\xi^P\cdot\xi^P)\xi^P\\
&\quad+\mu b_H^1b_H^2[(\xi^{2,H}\cdot\xi^P)(\xi^1\cdot\xi^2)(\xi^{1,H}\cdot\xi^P)+(\xi^{2,H}\cdot\xi^P)(\xi^2\cdot\xi^{1,H})(\xi^1\cdot\xi^P)\\
&\quad+(\xi^1\cdot\xi^P)(\xi^2\cdot\xi^P)(\xi^{1,H}\cdot\xi^{2,H})+(\xi^{1,H}\cdot\xi^P)(\xi^{2,H}\cdot\xi^P)(\xi^1\cdot\xi^2)\\
&\quad+(\xi^{1,H}\cdot\xi^P)(\xi^1\cdot\xi^{2,H})(\xi^2\cdot\xi^P)+(\xi^{1,H}\cdot\xi^{2,H})(\xi^2\cdot\xi^P)(\xi^1\cdot\xi^P)]\xi^P\\
&\quad+\frac{1}{4}Ab_H^1b_H^2[(\xi^{1,H}\cdot\xi^P)(\xi^1\cdot\xi^2)(\xi^{2,H}\cdot\xi^P)+(\xi^{1,H}\cdot\xi^P)(\xi^1\cdot\xi^{2,H})(\xi^2\cdot\xi^P)\\
&\quad+(\xi^1\cdot\xi^P)(\xi^{1,H}\cdot\xi^2)(\xi^{2,H}\cdot\xi^P)+(\xi^1\cdot\xi^P)(\xi^{1,H}\cdot\xi^{2,H})(\xi^2\cdot\xi^P)]\xi^P\\
&\quad+\frac{1}{4}Ab_H^1b_H^2[(\xi^{2,H}\cdot\xi^P)(\xi^1\cdot\xi^2)(\xi^{1,H}\cdot\xi^P)+(\xi^{2,H}\cdot\xi^P)(\xi^2\cdot\xi^{1,H})(\xi^1\cdot\xi^P)\\
&\quad+(\xi^2\cdot\xi^P)(\xi^{2,H}\cdot\xi^1)(\xi^{1,H}\cdot\xi^P)+(\xi^2\cdot\xi^P)(\xi^{2,H}\cdot\xi^{1,H})(\xi^1\cdot\xi^P)]\xi^P\\
&\quad+Bb_H^1b_H^2[(\xi^{1,H}\cdot\xi^{2,H})(\xi^1\cdot\xi^2)+(\xi^{1,H}\cdot\xi^2)(\xi^1\cdot\xi^{2,H})]\xi^P.
\end{aligned}
$$

Table 1 All possible nonlinear interactions

Interactions	Non-vanishing conditions
P+ P $\to$ SH	$\lambda + 2B + 3\mu + A \neq 0$
P + SH $\to$ P	$\lambda + 2B + 3\mu + A \neq 0$
P + SH $\to$ SH	$\lambda + 2\mu + \frac{1}{2}A + B \neq 0$ or $\mu + \frac{1}{2}A + B \neq 0$
P + SV$\to$ SV	$\lambda + B \neq 0$ or $2\mu + \frac{1}{2}A \neq 0$
SH + SH $\to$ P	Interaction condition and
	$\lambda + 2\mu + \frac{1}{2}A + B \neq 0$ or $\mu + \frac{1}{2}A + B \neq 0$
SH + SV $\to$ Ø	None
SV + SV $\to$ P	Interaction condition and
	$\lambda + B \neq 0$ or $2\mu + \frac{1}{2}A \neq 0$

SH stands for the S mode within the interaction plane, SV stands for the S mode perpendicular to the interaction plane. Non-vanishing condition means the principal symbols of the nonlinear responses $u^\bullet$ in Theorem 3 are non-vanishing. The interaction condition is in Lemma 3

Then we have

$$\begin{aligned}
\imath \mathbf{h}^{HH} &= b_H^1 b_H^2 |\xi^1||\xi^2| \Big[\lambda \cos^2\alpha + \mu(2\cos^2\alpha - \sin^2\alpha) + \frac{1}{2}A(\cos^2\alpha - \sin^2\alpha) \\
&\quad + B(\cos^2\alpha - \sin^2\alpha)\Big]\xi^P \\
&= b_H^1 b_H^2 |\xi^1||\xi^2| \Big[\Big(\lambda + 2\mu + B + \frac{1}{2}A\Big)\cos^2\alpha - \Big(\mu + B + \frac{1}{2}A\Big)\sin^2\alpha\Big]\xi^P.
\end{aligned}$$

This term is generically non-vanishing if $\lambda + 2\mu + \frac{1}{2}A + B \neq 0$ or $\mu + \frac{1}{2}A + B \neq 0$.

To conclude this section, we summarize all the possible interactions in Table 1.

5 The inverse problem

We complete the proofs of Theorem 1 in this section.

Proof of Theorem 1 First of all, from the displacement-to-traction map Λ, we derive the linearized map Λ_{lin} which corresponds to the linearized elastic equation (7), see (18). This problem have been studied in [19] for $\mu > 0$, $3\lambda + 2\mu > 0$ on $\overline{\Omega}$ and in a more general setting in [6] for $\mu > 0$, $\lambda + \mu > 0$. We conclude that one can determine the P and S wave speed $\sqrt{\lambda + 2\mu}$ and $\sqrt{\mu}$, hence λ and μ from Λ_{lin,T_0}, $T_0 > \mathrm{diam}_S(\overline{\Omega})$.

It is convenient to consider the P, SV wave interaction. So for any $(t_0, x_0) \in \mathbb{R} \times \Omega$ and ξ^1, ξ^2 two linearly independent vectors at x_0, we choose two geodesics $c_1(s), c_2(s)$ for $-dt^2 + g_P$, $-dt^2 + g_S$ respectively such that

$$c_\bullet(0) = (t_\bullet, x_\bullet) \in \mathbb{R} \times \partial\Omega, \quad c_\bullet(s_\bullet) = (t_0, x_0), \quad s_\bullet > 0,$$
$$\text{and } \dot{c}_\bullet(s_\bullet) = (\tau^\bullet, \xi^\bullet), \quad \bullet = 1, 2.$$

We let $\gamma^\bullet$ be the corresponding cotangent vectors at $(t_\bullet, x_\bullet)$, $\bullet = 1, 2$. Then we construct two distorted plane waves $u^{(1)}, u^{(2)}$ as in Definition 1 for γ^1, γ^2 and a small parameter δ. We take $f^{(1)} = f^{(1),P}$, $f^{(2)} = f^{(2),S}$ hence $u^{(1)} = u^{(1),P}$, $u^{(2)} = u^{(2),S}$ by Proposition 1. Following the nonlinear analysis in Sect. 4, we see that for δ sufficiently small, the nonlinear response u^{PSS} in Theorem 3 is a conormal distribution to Λ^{PSS} (away from the wave front sets of the linear responses). From the principal symbol of $\partial_{\epsilon_1}\partial_{\epsilon_2}\Lambda(f_\epsilon)|_{\epsilon_1=\epsilon_2=0}$ at the boundary (for a measurement time $T_0 > 2\mathrm{diam}_S(\Omega)$), we can determine the principal symbols of u^{PSS} at $\mathscr{Z}_{PS}$. From the symbol of the SV mode, we obtain the value of $\lambda + B$ and $2\mu + \frac{1}{2}A$ at x_0, see Table 1. This determines the value of A, B at x_0 and finishes the proof of Theorem 1.

Acknowledgements Maarten V. de Hoop acknowledges and sincerely thanks the Simons Foundation under the MATH+X program for financial support. He was also partially supported by the members of the Geo-Mathematical Imaging Group at Rice University and NSF under grant DMS-1815143. Gunther Uhlmann is partially supported by NSF and a Si-Yuan Professorship at HKUST.

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