SIAM J. APPL. MATH.
Vol. 79, No. 6, pp. 2359–2384

UNIQUE RECOVERY OF PIECEWISE ANALYTIC DENSITY AND STIFFNESS TENSOR FROM THE ELASTIC-WAVE DIRICHLET-TO-NEUMANN MAP*

MAARTEN V. DE HOOP†, GEN NAKAMURA‡, AND JIAN ZHAI§

Abstract. We study the recovery of piecewise analytic density and stiffness tensor of a three-dimensional domain from the local dynamical Dirichlet-to-Neumann map. We give global uniqueness results if the medium is (1) transversely isotropic with known axis of symmetry in each subdomain (2) orthorhombic with one of the three known symmetry planes tangential to a flat part of the accessible interface. We also obtain uniqueness of a fully anisotropic stiffness tensor, assuming that it is piecewise constant and that the interfaces which separate the subdomains have curved portions. The domain partition need not to be known. Precisely, we show that a domain partition consisting of subanalytic sets is simultaneously uniquely determined.

Key words. inverse boundary value problem, elastic waves, anisotropy

AMS subject classifications. 35R30, 35L10

DOI. 10.1137/18M1232802

1. Introduction.

We study the recovery of piecewise analytic density and stiffness tensors of a three-dimensional domain from the local dynamical Dirichlet-to-Neumann (DN) map. We introduce a domain partition and consider anisotropy and scattering off the interfaces separating the subdomains in the partition. This has been considered as an open problem in exploration seismology where anisotropy reveals critical information on Earth materials, microstructure in geological formations, and stress. The stress-induced anisotropy is analyzed in [21, 48].

We let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary $\partial\Omega$ and $y = (y^1, y^2, y^3)$ be Cartesian coordinates. We consider the initial boundary value problem for the system of equations describing elastic waves,

$$
\begin{cases}
\rho\partial_t^2 u = \operatorname{div}(\mathbf{C}\varepsilon(u)) =: Lu \quad \text{in } \Omega_T = \Omega \times (0,T), \\
u = f \quad \text{on } \partial\Omega \times (0,T), \\
u(y,0) = \partial_t u(y,0) = 0 \quad \text{in } \Omega,
\end{cases}
\tag{1.1}
$$

with $f(y,0) = 0$ and $\frac{\partial}{\partial t} f(y,0) = 0$ for $y \in \partial\Omega$. Here, $u = (u_1, u_2, u_3)$ denotes the displacement vector and

$$
\varepsilon(u) = (\nabla u + (\nabla u)^T)/2 = (\varepsilon_{ij}(u)) = \frac{1}{2}\left(\frac{\partial u_i}{\partial y^j} + \frac{\partial u_j}{\partial y^i}\right)
$$

*Received by the editors December 12, 2018; accepted for publication (in revised form) October 29, 2019; published electronically December 3, 2019.
https://doi.org/10.1137/18M1232802
Funding: The work of the first author was supported by the Simons Foundation under the MATH + X program, the National Science Foundation under grant DMS-1559587, and the corporate members of the Geo-Mathematical Group at Rice University. The work of the second author was supported by Grant-in-Aid for Scientific Research (15K21766 and 15H05740) of the Japan Society for the Promotion of Science (JSPS).
†Simons Chair in Computational and Applied Mathematics and Earth Science, Rice University, Houston, TX 77005 (mdehoop@rice.edu).
‡Department of Mathematics, Hokkaido University, Sapporo 060-0810, Japan (nakamuragenn@gmail.com).
§Institute for Advanced Study, The Hong Kong University of Science and Technology, Hong Kong, China (jian.zhai@outlook.com).

the linear strain tensor which is the symmetric part of ∇u in terms of the Cartesian coordinates. Furthermore, $\mathbf{C} = (C^{ijkl}) = (C^{ijkl}(y))$ is the stiffness tensor, and $\rho = \rho(y)$ is the density of mass, which are piecewise analytic on $\overline{\Omega}$.

Here, the hyperbolic or dynamical DN map Λ_T is given as the mapping

$$\Lambda_T : f \mapsto \partial_L u := (\mathbf{C}\varepsilon(u))\nu|_{\partial\Omega}, \tag{1.2}$$

where u is the solution of (1.1), $\mathbf{C}\varepsilon(u)$ is a 3×3 matrix with its (i, j) component $(\mathbf{C}\varepsilon(u))^{ij}$ given by $(\mathbf{C}\varepsilon(u))^{ij} = \sum_{k,l=1}^{3} C^{ijkl}\varepsilon_{kl}(u)$, and ν is the outward unit normal to $\partial\Omega$. Physically, $\partial_L u$ signifies the normal traction at $\partial\Omega$. Its mapping property, that is, the domain and target spaces, will be specified in section 2. Actually, we will consider a local DN map which is a localized version of the DN map. We are using the (full) DN map here just for simplicity.

It is physically natural to assume that ρ is bounded away from 0 on $\overline{\Omega}$ and that the stiffness tensor $\mathbf{C}$ satisfies the following symmetries and strong convexity condition:

- (symmetry) $C^{ijkl}(x) = C^{jikl}(x) = C^{klij}(x)$ for any $x \in \overline{\Omega}$ and i, j, k, l;
- (strong convexity) there exists a $\delta > 0$ such that for any 3×3 real-valued symmetric matrix (ε_{ij}),

$$\sum_{i,j,k,l=1}^{3} C^{ijkl}\varepsilon_{ij}\varepsilon_{kl} \geq \delta \sum_{i,j=1}^{3} \varepsilon_{ij}^2.$$

We first consider the following inverse boundary value problem: Can one determine C^{ijkl} and ρ (as well as all their derivatives) at the boundary from Λ_T? This inverse problem is referred to as the *boundary determination*. Concerning the uniqueness, this question was first answered by Rachelle [37] for the isotropic case, that is,

$$\mathbf{C} = \left(\lambda\delta^{ij}\delta^{kl} + \mu(\delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk})\right).$$

Her method depends on the decoupling of S- and P-waves on the boundary. This separation of polarizations, however, is not required for our proof.

There exist different techniques for showing the determination of coefficients of elliptic equations. One common way is to view the DN map (for some elliptic PDE) as a pseudodifferential operator and to recover the material parameters at the boundary from its symbol. This was first proposed by Sylvester and Uhlmann [45] for the equation describing electrostatics:

$$\nabla \cdot (\gamma\nabla u) = 0 \text{ with conductivity } 0 < \gamma \in C^\infty(\overline{\Omega}).$$

The DN map is defined by

$$\Lambda_\gamma : H^{1/2}(\partial\Omega) \ni \varphi \to \gamma\frac{\partial u}{\partial\nu} \in H^{-1/2}(\partial\Omega),$$

where u is the solution of the above equation with $u = \varphi$ on $\partial\Omega$. The symbol of Λ_γ simply has the leading order term (the principal symbol)

$$\sigma(\Lambda_\gamma)(x', \xi') = \gamma(x')|\xi'|.$$

Here $(x', \xi') \in T^*(\partial\Omega)$ and $|\xi'|$ is the length of the cotangent vector with respect to the metric on $\partial\Omega$ from the Euclidean metric of $\mathbb{R}^3$. It is almost immediate to recover γ from $\sigma(\Lambda_\gamma)$ [45]. The derivatives of γ can be recovered from the lower-order terms of the full symbol. For elastostatics, the reconstruction was given in [30, 31] for the isotropic case and in [33] for the transversely isotropic case. The same approach was also applied to (time-harmonic) Maxwell's equations [29, 44]. We remark here that the calculation of the principal symbol of the DN map for the elastic system is quite challenging.

In our previous paper [14], we showed that via a finite-time Laplace transform, we can reduce the dynamical problem to an elliptic one: determine the isotropic C^{ijkl} and ρ at the boundary from Λ^h, where Λ^h is the DN map corresponding to the elliptic system of equations

$$\begin{cases} \mathcal{M}v = \rho v - h^2 \operatorname{div}(\mathbf{C}\varepsilon(v)) = 0 \ \text{ in } \Omega, \\ v = \varphi \ \text{ on } \partial\Omega \end{cases} \tag{1.3}$$

with a parameter h which is the reciprocal of the Laplace variable $\tau > 0$, and Λ^h is defined by

$$\Lambda^h : H^{1/2}(\partial\Omega) \ni \varphi \mapsto h\partial_L v = h(\mathbf{C}\varepsilon(v))\nu|_{\partial\Omega} \in H^{-1/2}(\partial\Omega).$$

In exploration geophysics, full waveform inversion in Laplace domain has been explored in [41, 42]. In general, we do not have the exact Λ^h from Λ_T. However, if we view Λ^h as a semiclassical pseudodifferential operator with a small parameter h, we can recover the full symbol of Λ^h from Λ^T. Then we expect to reconstruct the material parameters at the boundary from the full symbol of Λ^h. We refer the reader to the book by Zworski [54] for an introduction to semiclassical pseudodifferential operators.

Generally, we believe that it is impossible to reconstruct a fully anisotropic elastic tensor from the dynamical DN map. However, there are some physically important symmetry restrictions—while allowing the presence of interfaces—that are more general than isotropy, on the stiffness tensor, under which we can still have an explicit reconstruction. For an introduction to these symmetries, we refer the reader to Tanuma [47] and Musgrave [17]. In this paper, we will first survey to what extent we can recover anisotropy.

We will give an explicit reconstruction formula of C^{ijkl} at part of the boundary $\Sigma \subset \partial\Omega$ from the semiclassical symbol of Λ^h if either of the following three conditions holds:

1. Σ is flat, and $\mathbf{C}$ is *transversely isotropic* (TI) with symmetry axis normal to Σ (*vertically transversely isotropic* (VTI));
2. Σ is flat, and $\mathbf{C}$ is *orthorhombic* with one of the three (known) symmetry planes tangential to Σ;
3. Σ is curved, and $\mathbf{C}$ and ρ are constant.

For more details, see Propositions 5.1 and 5.3 and Remark 5.6 below.

In elastostatics, Nakamura, Tanuma, and Uhlmann [33] gave an explicit reconstruction scheme for the transversely isotropic stiffness tensor assuming that the symmetry axis is tilted, that is, not normal to the boundary, while the information is not enough to recover the VTI case [32]. However, we can recover VTI elastic parameters from the semiclassical symbol of Λ^h. Generally speaking, this is because we have more information in dynamical data than in static data. We will give further explanation in section A.3.

For the *interior determination* from Λ_T with T large enough, uniqueness of smooth isotropic elastic tensor and density was shown under different geometrical conditions [38, 39, 46, 8]. We will study the *interior determination* of piecewise analytic parameters based on our *boundary determination* results. For elliptic equations, the *boundary determination* usually leads to the uniqueness of *interior determination* of piecewise analytic coefficients. Kohn and Vogelius [23] first established the relation in electrostatics. A recent paper by Cârstea, Honda, and Nakamura [10] gives a uniqueness theorem for piecewise constant stiffness tensors. The key in the proof is the continuation of the local elliptic DN map (see section 2 for the definition). If the coefficients of elliptic equations are discontinuous, a variational argument is convenient for this continuation. Ikehata [19] gave such an argument in order to construct the physical parameters in an inclusion. In [10], the authors adapted this variational argument for the continuation of the local elliptic DN map. Runge's approximation plays an important role in the continuation of data, which is in turn guaranteed by Holmgren's uniqueness theorem.

For our problem, we need to know the exact operator Λ^h, not only its full symbol. To get Λ^h, basically we need to have $\Lambda_{T'}$ for any T'. This is possible by time continuation of Λ_T, if T is large enough, and the assumption that $\mathbf{C}, \rho$ are piecewise analytic. Also, with the exact Λ^h, we can view it as a classical pseudodifferential operator. Under this classical setting, we can also recover tilted transversely isotropic (TTI) elastic parameters.

The time continuation is established with the boundary control (BC) method. We basically follow the steps sketched by Kurylev and Lassas [25]. The BC method was first introduced by Belishev [6]. Time-continuation procedure was established for the Lamé system in [7]. Essentially, we need $T > 2r$, where r is the approximate controllability time and will be given in Lemma 3.1. With the assumption of piecewise analyticity, the existence of the approximate controllability time is guaranteed by the unique continuation principle (UCP) for lateral Cauchy data, which is essentially the Holmgren–John uniqueness theorem. Indeed, relaxing the analyticity of the material parameters would require a very different method of proof. For acoustic wave equations, uniqueness results for the piecewise smooth case under geometric conditions can be found in [22, 9].

The key application of the problem we are considering is (reflection) seismology. Heterogeneous TTI elastic media are recognized as a realistic description of anisotropy in exploration seismic [12, 18] and in deep seismic sounding [1, 34]. Indeed, attempts to recover TTI media has a long history [20, 11]. In more recent years, with modern data acquisition, significant efforts have been put into three-dimensional applications [53, 3, 49, 51] still without an understanding of uniqueness of this inverse problem. Also, TI appears in nondestructive testing [27] and in medical elasticity imaging (elastography) [43]. On the one hand, as the major phases of Earth's upper mantle include olivine, one may argue that orthorhombic symmetry needs to be considered in inverse problems in seismology. Existing geophysical and geological data indicate that orthorhombic media with a horizontal symmetry plane should be rather common for naturally fractured subsurface reservoirs. Indeed, one of the most common reasons for orthorhombic anisotropy in sedimentary basins is a combination of parallel vertical cracks and vertical transverse isotropy in the reference medium [52, 40]. Initial estimation of orthorhombic elastic parameters dates back about two decades [2], while tilted symmetry planes were considered much more recently [28, 24]. Inverse problems in orthorhombic media remains a very active area of research in seismology [35].

In practical applications in seismology, the partitioning, as well as knowledge of symmetry axis and planes, is motivated by (structural) geology and tectonics.

Moreover, heterogeneities are typically captured by piecewise polynomial functions motivating the piecewise analyticity assumption.

The paper is organized in the following manner. In section 2, we show how to reduce the hyperbolic problem to an elliptic one and establish the relation between the dynamic Λ_T to the symbol of Λ^h. In section 3, we study the time continuation of Λ_T with piecewise analytic coefficients. In section 4, we introduce the boundary normal coordinates and obtain the symbol of Λ^h in these coordinates via a factorization of the operator $\mathcal{M}$ in (1.3). Finally, in section 5, we show the uniqueness of interior determination for the piecewise analytic material parameters.

2. Transformation to an elliptic problem. In this section, we show how to reduce the hyperbolic problem (1.1) to the elliptic problem (1.3). We will give a modified exposition of what is given in [14]. Throughout this section, we assume that $\mathbf{C}, \rho \in L^\infty(\Omega)$. We consider the local DN map. We introduce an open connected smooth part $\Sigma \subset \partial\Omega$.

For $r \geq 0$ we let $H^r_{co}(\Sigma)$ be the closure in $H^r(\Sigma)$ of the set

$$C^\infty_c(\Sigma) = \{f \in C^\infty(\partial\Omega) :\ \text{supp } f \subset \Sigma\}$$

and $H^{-r}(\Sigma)$ be its dual. We note that when $\Sigma = \partial\Omega$, $H^r_{co}(\Sigma) = H^r(\Sigma)$. Then we define the local DN map Λ^Σ_T by

$$\Lambda^\Sigma_T : C^2([0,T]; H^{1/2}_{co}(\Sigma)) \ni f \mapsto \mathbf{C}\varepsilon(u)\nu|_{\Sigma\times[0,T]} \in L^2([0,T]; H^{-1/2}(\Sigma)),$$

where u solves (1.1).

We also define the local DN map $\Lambda^{h,\Sigma}$ for the elliptic problem (1.3) by

$$\Lambda^{h,\Sigma} : H^{1/2}_{co}(\Sigma) \ni \varphi \mapsto h\mathbf{C}\varepsilon(v)\nu|_\Sigma \in H^{-1/2}(\Sigma),$$

where v solves (1.3). We let $\psi \in H^{1/2}_{co}(\Sigma)$, $\chi(t) = t^2$, and $f(x,t) = \chi(t)\psi(x)$. We take $u_0 \in H^1(\Omega)$ (by inverse trace theorem) such that $u_0 = \psi$ on $\partial\Omega$ and satisfies

$$\operatorname{div}(\mathbf{C}\varepsilon(u_0)) = 0 \ \text{ in } \Omega$$

with the estimate

$$\|u_0\|_{H^1(\Omega)} \leq C\|\psi\|_{H^{1/2}_{co}(\Sigma)}.$$

Then we seek a solution u of (1.1) in the form

$$u(y,t) = \chi(t)u_0(y) + u_1(y,t),$$

where $u_1(\cdot,t) \in L^2((0,T); H^1_0(\Omega))$ with

$$\partial_t u_1(\cdot,t) \in L^2((0,T); L^2(\Omega)), \partial^2_t u_1 \in L^2((0,T); H^{-1}(\Omega))$$

solves the following system in the weak sense:

$$\begin{cases} \rho\partial^2_t u_1 - \operatorname{div}(\mathbf{C}\varepsilon(u_1)) = F(y,t) \ \text{ in } \Omega_T = \Omega\times(0,T), \\ u_1 = 0 \ \text{ on } \Sigma = \partial\Omega\times(0,T), \\ u_1(y,0) = \partial_t u_1(y,0) = 0 \ \text{ in } \Omega, \end{cases} \tag{2.1}$$

where

$$F(y,t) = -2\rho u_0 \in L^2((0,T); L^2(\Omega)).$$

Problem (2.1) is equivalent to solving for $u_1 \in L^2((0,T); H_0^1(\Omega))$ with $\partial_t u_1(\cdot,t) \in L^2((0,T); L^2(\Omega))$, $\partial_t^2 u_1 \in L^2((0,T); H^{-1}(\Omega))$, which satisfies

$$-\int_0^T (\rho\partial_t u_1, \partial_t v)\, dt + \int_0^T B[u_1(\cdot,t), v(\cdot,t)]\, dt = \int_0^T (F(\cdot,t), v(\cdot,t))\, dt \tag{2.2}$$

for any $v \in C_0^\infty([0,T]; H_0^1(\Omega))$, where $(\cdot\,,\cdot)$ is the $L^2(\Omega)$ inner product,

$$B[\varphi,\psi] = \int_\Omega \mathbf{C}\varepsilon(\varphi) :: \varepsilon(\psi)\mathrm{d}y, \;\; \varphi,\psi \in H_0^1(\Omega),$$

and the notation :: denotes the inner product of matrices.

It is well known (cf. [26]) that there exists a unique solution $u_1 \in L^2((0,T); H_0^1(\Omega))$ of (2.2) with $\partial_t u_1 \in L^2((0,T); L^2(\Omega))$, $\partial_t^2 u_1 \in L^2((0,T); H^{-1}(\Omega))$. By possibly modifying the value of u_1 in a zero-measure set,

$$u_1 \in C^1([0,T], H^{-1}(\Omega)) \cap C^0([0,T], L^2(\Omega)),$$

while it satisfies the estimate

$$\|u_1(\cdot,t)\|_{L^2(\Omega)} + \|\partial_t u_1(\cdot,t)\|_{H^{-1}(\Omega)} \le C\|F\|_{L^2((0,T);L^2(\Omega))}, \;\; t \in [0,T] \tag{2.3}$$

with a constant $C > 0$ (see [15, 26] for the details of these). Therefore,

$$\int_0^T \partial_t^2 u_1(\cdot,t)e^{-\tau t}\mathrm{d}t = \tau^2 \int_0^T u_1(\cdot,t)e^{-\tau t}\mathrm{d}t + e^{-\tau T}(\partial_t u_1(\cdot,T) + \tau u_1(\cdot,T)) \;\; \text{in } H^{-1}(\Omega)$$

with the estimate

$$\|u_1(\cdot,T)\|_{L^2(\Omega)} + \|\partial_t u_1(\cdot,T)\|_{H^{-1}(\Omega)} \le CT^{1/2}\|\psi\|_{H_{co}^{1/2}(\Sigma)}.$$

Based on this observation, consider the finite-time Laplace transform $w(\cdot,\tau)$ of u_1:

$$w(\cdot,\tau) = \int_0^T u_1(\cdot,t)e^{-\tau t}\mathrm{d}t.$$

Then

$$(\rho\tau^2 w(\cdot,\tau), v) + B[w(\cdot,\tau), v] = (F_1, v), \;\; v \in H_0^1(\Omega)$$

with

$$F_1 := \int_0^T Fe^{-\tau t}\mathrm{d}t - e^{-\tau T}(\partial_t u_1(\cdot,T) + \tau u_1(\cdot,T)) \in H^{-1}(\Omega);$$

that is, w satisfies the elliptic equation

$$\begin{cases} \rho\tau^2 w - \operatorname{div}(\mathbf{C}\varepsilon(w)) = F_1, \\ w = 0 \;\; \text{on } \partial\Omega \end{cases}$$

in the weak sense.

Now let v satisfy (1.3) with $h = 1/\tau$ and the Dirichlet data φ taken as

$$\varphi = \psi\chi_1(\tau;T) \text{ with } \chi_1(\tau;T) = \int_0^T t^2 e^{-\tau t}\mathrm{d}t.$$

Then we will estimate

$$r(y,\tau) = v(y,\tau) - \int_0^T u(y,t)e^{-\tau t}\mathrm{d}t.$$

By a direct computation, χ_1 satisfies the estimate

$$\chi_1(\tau;T) \geq \frac{C}{\tau^3}$$

for some $C > 0$ independent of τ and T. Furthermore, $z = v - u_0\chi_1(\tau;T)$ satisfies

$$\begin{cases} \rho\tau^2 z - \operatorname{div}(\mathbf{C}\varepsilon(z)) = -\chi_1(\tau;T)\rho\tau^2 u_0, \\ z = 0 \quad \text{on } \partial\Omega. \end{cases}$$

We observe that $r = z - w$ and that it satisfies

$$\begin{cases} \rho\tau^2 r - \operatorname{div}(\mathbf{C}\varepsilon(r)) = e^{-\tau T}(\partial_t u_1(T) + \tau u_1(T)) + (2Te^{-\tau T} + \tau T^2 e^{-\tau T})u_0, \\ r = 0 \quad \text{on } \partial\Omega. \end{cases}$$

Then we have

$$\|r(\cdot,\tau)\|_{H^1(\Omega)} \leq C\tau T^3 e^{-\tau T}\|\psi\|_{H^{1/2}_{co}(\Sigma)}$$

with a constant C independent of τ and T by the standard elliptic regularity estimate.

Now consider the finite-time Laplace transform $\mathcal{L}_T u$ of u given as

$$\mathcal{L}_T u = \int_0^T u e^{-\tau t}\mathrm{d}t \text{ with } \tau > 0.$$

Then we have

$$\|\partial_L v - \mathcal{L}_T \partial_L u\|_{H^{-1/2}(\Sigma)} \leq C\tau T^3 e^{-\tau T}\|\psi\|_{H^{1/2}_{co}(\Sigma)}$$

or, equivalently,

$$\|\Lambda^{h,\Sigma}\varphi - h\mathcal{L}_T\Lambda^{\Sigma}_T\chi\chi_1^{-1}\varphi\|_{H^{-1/2}(\Sigma)} \leq C\left(\frac{T}{h}\right)^3 e^{-\frac{T}{h}}\|\varphi\|_{H^{1/2}_{co}(\Sigma)}.$$

Hence,

$$\|\Lambda^{h,\Sigma} - h\mathcal{L}_T\Lambda^{\Sigma}_T\chi\chi_1^{-1}\|_{H^{1/2}_{co}(\Sigma)\to H^{-1/2}(\Sigma)} \leq C\left(\frac{T}{h}\right)^3 e^{-\frac{T}{h}}, \tag{2.4}$$

which means that for a fixed $T > 0$,

$$\Lambda^{h,\Sigma} \sim h\mathcal{L}_T\Lambda^{\Sigma}_T\chi\chi_1^{-1},$$

which means that the both sides are equal modulo an operator mapping $H^{1/2}_{co}(\Sigma)$ to $H^{-1/2}(\Sigma)$ with estimates $\mathcal{O}(h^\infty)$. Thus, from $\mathcal{L}_T$, we can obtain the full symbol of $\Lambda^{h,\Sigma}$ viewed as a semiclassical pseudodifferential operator with a small parameter h.

We remark here that, in general, we could not have the full operator $\Lambda^{h,\Sigma}$ from Λ^{Σ}_T for a finite T, but we can get the full symbol of $\Lambda^{h,\Sigma}$, from which we can already expect to recover the material parameters at the boundary. Later, we will see in section 3 that we can get $\Lambda^{h,\Sigma}$ from $\Lambda^{\Sigma}_{T^*}$ for some T^* large enough, and the material parameters are piecewise analytic. This enables us to recover piecewise analytic densities and stiffness tensors.

3. Time continuation of the DN map. In this section, we show that we can obtain Λ_T^Σ for any $T > 0$ from $\Lambda_{T^*}^\Sigma$ for a fixed T^* large enough, assuming that the coefficients are piecewise analytic. We will follow [25].

We assume that Ω consists of a finite number of Lipschitz subdomains D_α, $\alpha = 1, \dots, K$. That is, $\overline{\Omega} = \cup_{i=1}^K \overline{D}_\alpha$, $D_\alpha \cap D_\beta = \emptyset$ if $\alpha \neq \beta$. We also assume that in each D_α, $\mathbf{C}$ and ρ are analytic up to its boundary. Since Ω is a domain, we can assume without loss of generality that there exist smooth nonempty $\Sigma_{\alpha+1} \subset \overline{D_\alpha} \cap \overline{D_{\alpha+1}}$, $\alpha = 1, \dots, K$ with $\Sigma = \Sigma_1 \subset \partial\Omega$. First, we prove the following global version of the Holmgren–John uniqueness theorem.

LEMMA 3.1. *There exists a finite $r > 0$ such that, for any $t \geq 2r$, if $e \in \mathcal{D}'((0,t) \times \Omega)$ satisfies*

$$\rho \partial_t^2 e = \operatorname{div}(\mathbf{C}\varepsilon(e)) \quad in\ \Omega \times (0,t),$$
$$e|_{\Sigma \times [0,t]} = (\mathbf{C}\varepsilon(e))\nu|_{\Sigma \times [0,t]} = 0,$$

then $e(\frac{t}{2}) = \partial_t e(\frac{t}{2}) = 0$ in Ω.

Here and in the remainder of this section we will suppress the space coordinates in our notation.

Proof. First, we have by the standard Holmgren theorem (cf. [50]) that e vanishes on $D_1 \times [\varrho_1, t - \varrho_1]$ for some $\varrho_1 > 0$ if $\frac{t}{2} > \varrho_1$. For the unique continuation across the interfaces, we follow the reasoning in [36] in the following argument. First, we apply an analytic continuation of $\mathbf{C}, \rho$ on D_2 to a small neighborhood U_2 of Σ_2. Use $\mathbf{C}_{D_2}, \rho_{D_2}$ to denote the extended coefficients on $\tilde{D}_2 = D_2 \cup U_2$. Now, e satisfies

$$\rho_{D_2} \partial_t^2 e = \operatorname{div}(\mathbf{C}_{D_2}\varepsilon(e)) \ \text{ in } \tilde{D}_2 \times (\varrho_1, t - \varrho_1).$$

Since e vanishes on $\tilde{D}_2 \cap D_1$, we can apply Holmgren's theorem again to conclude that e vanishes on $D_2 \times (\varrho_1 + \varrho_2, t - \varrho_1 - \varrho_2)$ for some $\varrho_2 > 0$. We need $\frac{t}{2} > \varrho_1 + \varrho_2$. We can repeat the process and prove the lemma provided that r is sufficiently large. ◻

Let u^f be the solution of (1.1) with boundary value f.

LEMMA 3.2. *The pairs $(u^f(2r), -\partial_t u^f(2r))$, $f \in C_c^\infty(\Sigma \times (0, 2r))$ are dense in $H_0^1(\Omega) \times L^2(\Omega)$.*

Proof. Assume that a pair

$$(\alpha, \beta) \in (H_0^1(\Omega) \times L^2(\Omega))' = H^{-1}(\Omega) \times L^2(\Omega)$$

satisfies

$$\langle \alpha, u^f(2r) \rangle_{(H^{-1}(\Omega), H_0^1(\Omega))} + \langle \beta, -\partial_t u^f(2r) \rangle_{L^2(\Omega)} = 0$$

for all $f \in C_c^\infty(\Sigma \times (0, 2r))$, where $\langle \cdot\,, \cdot \rangle_{(H^{-1}(\Omega), H_0^1(\Omega))}$ is a pairing between an element in $H^{-1}(\Omega)$ and an element in $H^1(\Omega)$ defined as a continuous extension of the $L^2(\Omega)$ inner product. It is sufficient to show that

$$\alpha = \beta = 0.$$

Let e be the unique solution of

$$\begin{cases} \rho \partial_t^2 e = \operatorname{div}(\mathbf{C}\varepsilon(e)) \ \text{ in } \Omega \times (0, 2r), \\ e = 0 \ \text{ on } \partial\Omega \times (0, 2r), \\ \rho e(y, 2r) = \beta, \ \rho \partial_t e(y, 2r) = \alpha \ \text{ in } \Omega \end{cases} \tag{3.1}$$

with

$$e \in C([0,2r];L^2(\Omega)),\ \ \partial_t e \in C([0,2r];H^{-1}(\Omega)).$$

We note that the well-posedness of the above problem was established in [26].

Upon integration by parts, we obtain

$$\begin{aligned}0 &= \int_0^{2r} \big(\langle(\rho\partial_t^2 e - \mathrm{div}(\mathbf{C}\varepsilon(e)), u^f\rangle_{(H^{-1}(\Omega),H_0^1(\Omega))} \\ &\qquad - \langle(\rho\partial_t^2 u^f - \mathrm{div}(\mathbf{C}\varepsilon(u^f))), e\rangle_{(H^{-1}(\Omega),H_0^1(\Omega))}\big)\mathrm{d}t \\ &= \langle\beta, \partial_t u^f(2r)\rangle_{L^2(\Omega)} - \langle\alpha, u^f(2r)\rangle_{(H^{-1}(\Omega),H_0^1(\Omega))} \\ &\quad - \int_0^{2r} \langle(\mathbf{C}\varepsilon(e))\nu, f\rangle_{(H^{-1/2}(\Sigma),H^{1/2}(\Sigma))}\mathrm{d}t \\ &= -\int_0^{2r} \langle(\mathbf{C}\varepsilon(e))\nu, f\rangle_{(H^{-1/2}(\Sigma),H^{1/2}(\Sigma))}\mathrm{d}t\end{aligned}$$

for any $f \in C_c^\infty(\Sigma\times(0,2r))$, where $\langle\,\cdot\,,\,\cdot\,\rangle_{(H^{-1/2}(\Omega),H^{1/2}(\Omega))}$ is defined likewise $\langle\,\cdot\,,\,\cdot\,\rangle_{(H^{-1}(\Omega),H_0^1(\Omega))}$. Hence, we have

$$\int_0^{2r} \langle(\mathbf{C}\varepsilon(e))\nu, f\rangle_{(H^{-1/2}(\Sigma),H^{1/2}(\Sigma))}\mathrm{d}t = 0.$$

This yields

$$e|_{\Sigma\times[0,2r]} = (\mathbf{C}\varepsilon(e))\nu|_{\Sigma\times[0,2r]} = 0.$$

By Lemma 3.1, we have

$$e(r) = \partial_t e(r) = 0 \ \text{ on } \Omega.$$

Thus, $e = 0$ on $\Omega\times[0,2r]$, and hence $\alpha = \beta = 0$. ∎

We consider a bilinear form

$$E(u^f,u^g,t) = \int_\Omega \rho\partial_t u^f(t)\partial_t u^g(t) + \mathbf{C}\varepsilon(u^f(t)) :: \varepsilon(u^g(t))\mathrm{d}y.$$

To simplify the notation, we write $E(u^f,t) = E(u^f,u^f,t)$.

Lemma 3.3. *The operator* Λ_t^Σ *determines* $E(u^f,u^g,t)$ *for* $f,g \in C_c^\infty(\Sigma\times(0,t))$.

We remark there that the time T^* is exactly what is needed at the beginning of this section.

Proof. By the estimates for $\partial_t u^f$,$\partial_t^2 u^f$, we have

$$u^f \in C^1([0,t];H_0^1(\Omega)) \cap C^2([0,t];L^2(\Omega)).$$

Integrating by parts, we find that

$$\begin{aligned}\partial_t E(u^f,t) &= 2\int_\Omega \rho\partial_t^2 u^f(t)\partial_t u^f(t) + \mathbf{C}\varepsilon(u^f(t)) :: \varepsilon(\partial_t u^f(t))\mathrm{d}y \\ &= 2\langle\mathbf{C}\varepsilon(u^f(t))\nu, \partial_t u^f(t)\rangle_{(H^{-1/2}(\Sigma),H_{co}^{1/2}(\Sigma))} \\ &= 2\langle(\Lambda_t^\Sigma f)(t), \partial_t f(t)\rangle_{(H^{-1/2}(\Sigma),H_{co}^{1/2}(\Sigma))}.\end{aligned}$$

With the initial conditions, $E(u^f,0) = 0$, and we can determine $E(u^f,t)$ as well as

$$E(u^f,u^g,t) = \frac{1}{4}(E(u^{f+g},t) - E(u^{f-g},t))$$

by polarization. ∎

We arrive at the following theorem.

THEOREM 3.4. *Let $T^* > 2r$; then $\Lambda^\Sigma_{T^*}$ determines Λ^Σ_T for any $T > 0$.*

Proof. Let $\delta = \frac{T^*-2r}{2}$. It is sufficient to show that $\Lambda^\Sigma_{T^*}$ determines $\Lambda^\Sigma_{T^*+\delta}$. Indeed, by repeating the process presented below, we obtain the result.

For any $f \in C^\infty([0, T^*+\delta], H^{1/2}_{co}(\Sigma))$, take a decomposition $f = g + h$, where $g \in C^\infty_c([0, 2\delta), H^{1/2}_{co}(\Sigma))$ and $h \in C^\infty_c((\delta, T^*+\delta], H^{1/2}_{co}(\Sigma))$. Since we have $(\Lambda^\Sigma_{T^*+\delta} h)(t) = (\Lambda^\Sigma_{T^*} Y_{-\delta} h)(t)$ with $Y_{-\delta} h(t) := h(t+\delta)$ for $t \in [0, T^*]$ and

$$\Lambda^\Sigma_{T^*+\delta} f = \Lambda^\Sigma_{T^*+\delta} g + \Lambda^\Sigma_{T^*+\delta} h,$$

we only need to show that $\Lambda^\Sigma_{T^*}$ determines $(\Lambda^\Sigma_{T^*+\delta} g)(t)$ for any $t \in (T^*, T^*+\delta]$.

Let $t_0 = 2r + \delta$. By Lemma 3.2, there are $g_n \in C^\infty_c(\Sigma \times (0, 2r))$ such that

$$\lim_{n\to\infty} (u^{g_n}(2r), \partial_t u^{g_n}(2r)) = (u^g(t_0), \partial_t u^g(t_0)) \tag{3.2}$$

in the $H^1_0(\Omega) \times L^2(\Omega)$ topology. It is straightforward to show that (3.2) is equivalent to

$$\lim_{n\to\infty} E(u^{\tilde{g}_n}, t_0) = 0. \tag{3.3}$$

Here, $\tilde{g}_n(t) = g(t) - g_n(t-\delta)$ with $g_n(s) = 0, -\delta < s < 0$. By Lemma 3.3, we can use only $\Lambda^\Sigma_{T^*}$ to construct g_n satisfying (3.2). The functions $y_n(t) := u^{g_n}(t)$ for $t \in [2r, T^*]$ are the solutions of the initial boundary value problem:

$$\rho \partial_t^2 y_n = \operatorname{div}(\mathbf{C}\varepsilon(y_n) \quad \text{in } \Omega \times [2r, T^*],$$
$$y_n|_{\partial\Omega\times[2r,T^*]} = 0,\ \ y_n(2r) = u^{g_n}(2r), \quad \partial_t y_n(2r) = \partial_t u^{g_n}(2r).$$

We note that $y(t) := u^g(t+\delta)$ satisfies the same equation with initial data

$$y(2r) = u^g(t_0), \quad \partial_t y(2r) = \partial_t u^g(t_0).$$

Also by the continuous dependence of solutions on initial data, we have

$$\lim_{n\to\infty} \mathbf{C}\varepsilon(y_n)\nu|_{\Sigma\times[2r,T^*]} = \mathbf{C}\varepsilon(y)\nu|_{\Sigma\times[2r,T^*]}$$

in the L^2 topology. Hence,

$$\begin{aligned} (\Lambda^\Sigma_{T^*+\delta} g)(t) &= C\epsilon(u^g(t))\nu|_{\Sigma\times[t_0,T^*+\delta]} = \mathbf{C}\varepsilon(y(t-\delta))\nu|_{\Sigma\times[2r,T^*]} \\ &= \lim_{n\to\infty} \mathbf{C}\varepsilon(y_n(t-\delta))\nu|_{\Sigma\times[2r,T^*]} = \lim_{n\to\infty} (\Lambda^\Sigma_{T^*} Y_\delta\, y_n)(t) \\ &= \left(Y_{-\delta}\left(\lim_{n\to\infty} \Lambda^\Sigma_{T^*} g_n\right)\right)(t) \quad \text{for } t \in [t_0, T^*+\delta], \end{aligned} \tag{3.4}$$

and we can determine $(\Lambda^\Sigma_{T^*+\delta} g)(t)$ on $[T^*, T^*+\delta]$ from $\Lambda^\Sigma_{T^*} g_n$. ∎

4. The principal symbol of $\Lambda^{h,\Sigma}$. The principal symbol of $\Lambda^{h,\Sigma}$ as a semiclassical pseudodifferential operator is analyzed in [14]. For the self-containedness of this article, we will sketch the key points in the following.

For the analysis, we need to introduce the boundary normal coordinates. Given a boundary point $p_0 \in \Sigma$, let $(x^1(p'), x^2(p'))$ be local coordinates of Σ close to p_0. For any p near p_0, we use the boundary normal coordinates $x(p) = (x^1(p'), x^2(p'), x^3)$, where p' is the nearest point on Σ to p with $x(p') = (x^1(p), x^2(p), 0)$ and $x^3 = \mathrm{dist}(p, p')$. Here the distance function $\mathrm{dist}(\cdot,\cdot)$ is with respect to the Euclidean metric. Thus, Σ and Ω are locally represented by $x^3 = 0$ and $x^3 > 0$ near p_0. We let (ξ_1, ξ_2, ξ_3) and (η_1, η_2, η_3) represent the same conormal vector with respect to different coordinates, (x^1, x^2, x^3) and (y^1, y^2, y^3), such that $\xi_\alpha \mathrm{d}x^\alpha = \eta_i \mathrm{d}y^i$ using the Einstein summation convention, which will be repeatedly used in the paper. Here y denotes the Cartesian coordinates introduced before. We introduce the coordinate mapping F as

$$F(y^1(p), y^2(p), y^3(p)) = (x^1(p), x^2(p), x^3(p))$$

and the Jacobian of F as

$$J^a_{\ i} = \left(\frac{\partial x^a}{\partial y^i}\right). \tag{4.1}$$

Then $J^a_{\ i}\xi_a = \eta_i$ (or, equivalently, $J^T\xi = \eta$) and

$$\tilde{C}^{abcd}(p) = J^a_{\ i}J^b_{\ j}J^c_{\ k}J^d_{\ l}C^{ijkl}(p) \quad \text{and} \quad G^{ab} = J^a_{\ i}J^b_{\ j}\delta^{ij} =: G^{-1}.$$

Here, $G = (G_{ab})$ is the induced Riemannian metric for boundary normal coordinates, x. Also,

$$J^a_{\ i}\tilde{v}_a = v_i.$$

In the boundary normal coordinates, (1.3) attains the form

$$\begin{cases} (\tilde{\mathcal{M}}\tilde{v})^a = \rho G^{ac}\tilde{v}_c - h^2 \displaystyle\sum_{b,c,d=1}^{3} \nabla_b(\tilde{C}^{abcd}\varepsilon_{cd}(\tilde{v})) = 0 \text{ in } \{x^3 > 0\} \text{ for } 1 \le a \le 3, \\ \tilde{v}_d|_{x^3=0} = \tilde{\psi}_d, \quad 1 \le d \le 3, \end{cases} \tag{4.2}$$

where ∇_a is the covariant derivative with respect to metric G and $\varepsilon_{cd}(\tilde{v}) = \frac{1}{2}(\nabla_c\tilde{v}_d + \nabla_d\tilde{v}_c)$.

We express $\Lambda^{h,\Sigma}$ in boundary normal coordinates as

$$\Lambda^{h,\Sigma} : \tilde{\psi}_d \to -h\tilde{C}^{a3cd}\varepsilon_{cd}(\tilde{v})|_\Sigma.$$

Here we denote $\xi = (\xi', \xi_3) = (\xi_1, \xi_2, \xi_3)$. Then $\Lambda^{h,\Sigma}$ is a semiclassical pseudodifferential operator with full symbol $\tilde{\sigma}(\Lambda^{h,\Sigma})(x', \xi')$ which has the asymptotics [14]

$$\tilde{\sigma}(\Lambda^{h,\Sigma})(x', \xi') = \sum_{j \ge 0} h^j \lambda_{-j}(x', \xi').$$

In this expansion, $\lambda_0(x', \xi')$ signifies the principal symbol of $\Lambda^{h,\Sigma}$. We proceed with calculating $\lambda_0(x', \xi')$.

We define

$$\begin{aligned} \tilde{Q}(x, \xi') &= \left(\sum_{b,d=1}^{2} \tilde{C}^{abcd}(x)\xi_b\xi_d;\ 1 \le a,\, c \le 3\right), \\ \tilde{R}(x, \xi') &= \left(\sum_{b=1}^{2} \tilde{C}^{abc3}(x)\xi_b;\ 1 \le a,\, c \le 3\right), \\ \tilde{D}(x) &= \left(\tilde{C}^{a3c3}(x);\ 1 \le a,\, c \le 3\right), \end{aligned} \tag{4.3}$$

and then

$$\tilde{M}(x,\xi) = \tilde{D}(x)\xi_3^2 + (\tilde{R}(x,\xi') + \tilde{R}^T(x,\xi'))\xi_3 + \tilde{Q}(x,\xi') + \rho(x)G \tag{4.4}$$

is the principal symbol of $\tilde{\mathcal{M}}$. First, we introduce the following factorization of $\tilde{M}$.

We note that $\tilde{M}(x,\xi)$ is a positive definite matrix for $x \in \overline{\Omega}$, $\xi \in \mathbb{R}^3\backslash 0$. Hence, for fixed (x,ξ'), $\det \tilde{D}^{-1/2}\tilde{M}(x,\xi)\tilde{D}^{-1/2} = 0$ in ξ_3 admits 3 roots $\xi_3 = \zeta_j$ $(j = 1,2,3)$ with positive imaginary parts and 3 roots $\overline{\zeta_j}$ $(j = 1,2,3)$ with negative imaginary parts. Thus, we can state the following.

Lemma 4.1 ([16]). *There is a unique factorization*

$$\check{M}(x,\xi) = \tilde{D}(x)^{-1/2}\tilde{M}(x,\xi)\tilde{D}(x)^{-1/2} = (\xi_3 - \check{S}_0^*(x,\xi'))(\xi_3 - \check{S}_0(x,\xi'))$$

with $\mathrm{Spec}(\check{S}_0(x,\xi')) \subset \mathbb{C}_+$, *where* $\mathrm{Spec}(\check{S}_0(x,\xi'))$ *is the spectrum of* $\check{S}_0(x,\xi')$. *In the above,*

$$\check{S}_0(x,\xi') := \left(\oint_\gamma \zeta\check{M}(x,\xi',\zeta)^{-1}\mathrm{d}\zeta\right)\left(\oint_\gamma \check{M}(x,\xi',\zeta)^{-1}\mathrm{d}\zeta\right)^{-1},$$

where $\gamma \subset \mathbb{C}_+ := \{\zeta \in \mathbb{C} : \mathrm{Im}\,\zeta := \text{imaginary part of } \zeta > 0\}$ *is a continuous curve enclosing all the roots* ζ_j $(j = 1,2,3)$ *of* $\det(\check{M}(x,\xi',\zeta)) = 0$ *in* $\zeta \in \mathbb{C}_+$.

This lemma implies the following factorization of $\tilde{M}(x,\xi)$:

$$\tilde{M}(x,\xi) = (\xi_3 - \tilde{S}_0^*(x,\xi'))\tilde{D}(x)(\xi_3 - \tilde{S}_0(x,\xi')), \tag{4.5}$$

where

$$\tilde{S}_0(x,\xi') = \tilde{D}^{-1/2}(x)\check{S}_0(x,\xi')\tilde{D}^{1/2}(x).$$

We have ([14, Proposition 3.3])

$$\lambda_0(x',\xi') = -\mathrm{i}(\tilde{D}(x',0)\tilde{S}_0(x',0,\xi') + \tilde{R}(x',0,\xi')). \tag{4.6}$$

Roughly speaking, this is because $\lambda_0(x',\xi') \sim -\mathrm{i}(\tilde{D}(x',0)\xi_3 + \tilde{R}(x',0,\xi'))$ and $\xi_3 \sim \tilde{S}_0(x',0,\xi')$.

By [14, Proposition 3.4], we obtain $D^\alpha_{x^3}\lambda_0$ for any $\alpha = 1,2,\ldots,$ from the lower-order terms λ_{-j}, $j = 1,2,\ldots,$ of the full symbol of $\Lambda^{h,\Sigma}$. In order to give an explicit reconstruction of the material parameters at the boundary, we need to calculate the closed form of $\lambda_0(x',\xi')$.

Surface impedance tensor. We need to have the explicit closed form of the principal symbol λ_0. The calculation in boundary normal coordinates would be extremely complicated. In this part, we establish the relation between the principal symbol λ_0, which is defined in boundary normal coordinates x, and the so-called surface impedance tensor Z, which is defined in Cartesian coordinates y. A similar discussion can be found in [14, section 4].

Take $\mathbf{n}$ to be the outer normal direction at $p \in \Sigma$ expressed in Cartesian coordinate. Denote $\mathbf{n} = (n_1,n_2,n_3)$. Let $\mathbf{m} = (m_1,m_2,m_3)$ be a vector (not a unit one) normal to $\mathbf{n}$. We have

$$J^{-T}\mathbf{n}(p) = (0,0,1),\ J^{-T}\mathbf{m}(p) = (\xi'(p),0) = (\xi_1(p),\xi_2(p),0) \tag{4.7}$$

with the Jacobian $J = (J^a_i)$, defined in (4.1), at p.

The operator $\mathcal{M}$ in (1.3) has principal symbol $M = (M^{ik}(p,\eta))$ at p given by

$$M^{ik}(p,\eta) = \sum_{j,l=1}^{3} C^{ijkl}(p)\eta_j\eta_l + \rho\delta^{ik}$$

in y coordinates, and operator $\tilde{\mathcal{M}}$ in (4.2) has principal symbol $\tilde{M}$

$$\tilde{M}^{ac}(p,\xi) = \sum_{b,d=1}^{3} \tilde{C}^{abcd}(p)\xi_b\xi_d + \rho G^{ac}$$

in x coordinates. Using the transformation rules of tensors, we have

$$J^a_{\ i} M^{ik}(p,\eta) J^c_{\ k} = J^a_{\ i} J^b_{\ j} J^c_{\ k} J^d_{\ l} C^{ijkl}(p)\xi_b\xi_d + J^a_{\ i} J^c_{\ k}\rho\delta^{ik},$$

which is nothing but

$$JMJ^T = \tilde{M}.$$

We choose $\eta = q\mathbf{n}+\mathbf{m} = (qn_1+m_1, qn_2+m_2, qn_3+m_3)$ so that $\xi = J^{-T}(q\mathbf{n}+\mathbf{m}) = (\xi_1, \xi_2, q)$. It follows that

$$J^{-1}\tilde{M}(p,\xi)J^{-T} = M(p, q\mathbf{n}+\mathbf{m}).$$

We obtain

$$M(p, q\mathbf{n}+\mathbf{m}) = Dq^2 + \left(R + R^T\right) q + Q + \rho$$

with

$$\begin{aligned} D(\mathbf{n}) &= \left(\sum_{j,l=1}^{3} C^{ijkl} n_j n_l; 1 \le i,\ k\ \le 3\right), \\ R(\mathbf{n},\mathbf{m}) &= \left(\sum_{j,l=1}^{3} C^{ijkl} m_j n_l; 1 \le i,\ k\ \le 3\right), \\ Q(\mathbf{m}) &= \left(\sum_{j,l=1}^{3} C^{ijkl} m_j m_l; 1 \le i,\ k\ \le 3\right). \end{aligned} \tag{4.8}$$

Similar to Lemma 4.1, there is a unique factorization of M, that is,

$$M(p, q\mathbf{n}+\mathbf{m}) = \left(q - S_0^*\right) D(q - S_0), \quad \operatorname{Spec}\left(S_0(\mathbf{n},\mathbf{m})\right) \subset \mathbb{C}_+,$$

where $S_0(\mathbf{n},\mathbf{m})$ is independent of q. Changing coordinates,

$$\begin{aligned} \tilde{M}(p,\xi) &= JMJ^T = J(q - S_0^*)D(q - S_0)J^T \\ &= \left(q - JS_0^*J^{-1}\right)\left(JDJ^T\right)\left(q - J^{-T}S_0J^T\right). \end{aligned}$$

Hence, by the fact $Spec(J^{-T}S_0J^T) \subset \mathbb{C}_+$ and the uniqueness of the factorization,

$$\tilde{S}_0 = J^{-T}S_0J^T.$$

We define the surface impedance tensor $Z = Z(p,\mathbf{m},\mathbf{n})$ by

$$Z(p,\mathbf{m},\mathbf{n}) = -\mathrm{i}(DS_0 + R^T).$$

Based on the previous arguments, we can now express the principal symbol λ_0 in terms of Z:

LEMMA 4.2. *The principal symbol $\lambda_0(x'(p), \xi')$ is related to the surface impedance tensor as*

$$\lambda_0(x'(p), \xi') = JZ(p, \mathbf{m}, \mathbf{n})J^T, \tag{4.9}$$

where the relation between ξ' and $\mathbf{n}, \mathbf{m}$ is defined in (4.7).

The reconstruction of the density and stiffness tensor for the principal symbol is now simplified to a reconstruction from the surface impedance tensor. The derivatives of Z can be computed from the derivatives of λ_0 since the matrix J is independent of the elastic parameters. The recovery of elastic parameters from Z and its derivatives are given in the appendix.

5. Recovery of the material parameters.

5.1. Recovery at the boundary. In this subsection, we summarize our results on recovering of stiffness tensor and the density at the boundary from Λ_T^Σ. We only need to recover from the surface impedance tensor Z for the elliptic problem introduced above. We emphasize that in this subsection, we can take $T > 0$ giving the time interval $(0, T)$.

PROPOSITION 5.1. *Assume that Σ is flat. For the following cases, the local DN map Λ_T^Σ identifies $(\mathbf{C}, \rho)$ and all their derivatives on Σ uniquely. There is an explicit reconstruction procedure for these identifications:*

1. *The stiffness tensor $\mathbf{C}$ is transversely isotropic in a neighborhood of Σ with the symmetry axis normal to Σ.*
2. *The stiffness tensor $\mathbf{C}$ is orthorhombic in a neighborhood of Σ with one of the three (known) symmetry planes tangential to Σ.*

The explicit reconstruction procedure for the parameters from the surface impedance tensor can be found in Appendix A.

Remark 5.2. For the transversely isotropic case that the symmetric axis is nowhere normal to Σ we can also obtain the reconstruction if T is large enough. See Proposition 5.4 in the next section.

PROPOSITION 5.3. *Assume that the stiffness tensor $\mathbf{C}$ is homogeneous in a neighborhood of Σ and Σ has a curved part. The local DN map Λ_T^Σ identifies $(\mathbf{C}, \rho)$ in this neighborhood uniquely.*

The proof of the above proposition can be found in Appendix B.

5.2. Recovery in the interior. We finally consider the recovery of a piecewise analytic density and stiffness tensor in the interior of the domain. We begin with estimate (2.4), leading to

$$\Lambda^{h,\Sigma} = \lim_{T\to\infty} h\mathcal{L}_T \Lambda_T^\Sigma \chi\chi_1^{-1}(h; T).$$

Therefore, we can consider the fully elliptic problem if we have $\Lambda_{T^*}^\Sigma$, where $T^* > 2r$ with r defined in Lemma 3.1 as the data and adapt the procedure in [10] to study the problem of recovering piecewise analytic material parameters. Once we have the boundary determination at Σ, by analyticity of the coefficients in subdomain D_1, we can propagate the data to the interior interface Σ_2 and iterate the boundary determination results. We will sketch the procedure below in detail.

Now, we have the exact elliptic local DN map $\Lambda^{h,\Sigma}$ from $\Lambda_{T^*}^\Sigma$. For the TI case, if the symmetry axis is normal to Σ, we can recover the parameters on the boundary from the semiclassical symbol of $\Lambda^{h,\Sigma}$. However, if the symmetry axis is not normal to Σ, this approach would fail. Then we consider $\Lambda^{h,\Sigma}$ as a classical pseudodifferential operator and adapt the procedure developed in [33] for their reconstruction. (We can always reduce to the above two situations by possibly passing to further subset of Σ.)

PROPOSITION 5.4. *Assume that* $\mathbf{C}$ *is smooth and transversely isotropic with symmetry axis nowhere normal to* Σ*, and assume that* ρ *is smooth. Then we have an explicit reconstruction of* ρ *and* $\mathbf{C}$*, as well as their derivatives on* Σ*, from the symbols of* $\Lambda^{h_1,\Sigma}$ *and* $\Lambda^{h_2,\Sigma}$*,* $h_1 \neq h_2$*, considered as classical pseudodifferential operators.*

We note here that if the symmetry axis of TI is nowhere normal to Σ, then Σ does not contain an open flat part.

With all the boundary determination results developed before, we are ready to have the uniqueness for interior determination of piecewise analytic parameters. We assume the domain partitioning introduced in section 3 throughout this section. First, by applying the boundary determination results of Propositions 5.1, 5.3, and 5.4 to Σ, we can recover the jets of the elastic modulus (i.e., the modulus and their derivatives) on Σ under different assumptions for those propositions. If the parameters are analytic on D_1 and $\Sigma \subset \partial D_1$, we have already recovered the parameters in D_1. We summarize the results in the following proposition.

PROPOSITION 5.5. *If* $\Lambda^{h,\Sigma}_{\mathbf{C}_1,\rho_1} = \Lambda^{h,\Sigma}_{\mathbf{C}_2,\rho_2}$ *and on* D_1*,* $\mathbf{C}_j, \rho_j$*,* $j = 1, 2$ *are analytic and one of the following conditions holds:*

1. $\mathbf{C}_j$ *are TI with a known symmetry axis; that is, there exist Cartesian coordinates* y *in* D_1 *such that the nonzero components of* $\mathbf{C}_j(y)$ *are those listed in Appendix* A;
2. Σ *is (partly) flat, and* $\mathbf{C}_j$ *are orthorhombic with one of the three (known) symmetry planes tangential to* Σ*; that is, there exist Cartesian coordinates* y *in* D_1 *such that the nonzero components of* $\mathbf{C}_j(y)$ *are those listed in Appendix* A;
3. Σ *is (partly) curved, and* $\mathbf{C}_j, \rho_j$*,* $j = 1, 2$ *are constant in* D_1*;*

then $\mathbf{C}_1 = \mathbf{C}_2$*,* $\rho_1 = \rho_2$ *on* D_1*.*

Remark 5.6. For determination of TI parameters, we apply (1) Proposition 5.1 if Σ is flat and the symmetry axis is normal to Σ and recover the jets of modulus or (2) Proposition 5.4 if otherwise and recover on some part of Σ which the symmetry axis is nowhere normal to. For orthorhombic modulus, we only have boundary determination results if (part of) Σ is flat and one of the three known symmetry planes is tangential to Σ.

In order to use the boundary determination results to have the uniqueness in the interior, we need the propagation of the DN map. Let $D_\beta, \beta = 1, 2, \dots, \alpha$ be a chain of subdomains of Ω such that $\Sigma_1 := \Sigma \subset \partial D_1$. Here the chain of subdomains D_β's means that it satisfies the following conditions: (i) $D_\beta \cap D_{\beta'} = \emptyset$ if $\beta \neq \beta'$, and (ii) there are nonempty smooth surfaces $\Sigma_\beta \subset \overline{D_\beta} \cap \overline{D_{\beta+1}}$, $\beta = 1, 2, \dots, \alpha - 1$. Further let $\Omega_\alpha = \Omega \setminus \cup_{\beta=1}^{\alpha-1} \overline{D_\beta}$ and $\Sigma_\alpha \subset \partial\Omega_\alpha$ be open, connected, and smooth. Define $\Lambda^{h,\Sigma_\alpha}_{\mathbf{C},\rho}$ similar to $\Lambda^{h,\Sigma}_{\mathbf{C},\rho}$ with (Ω, Σ) replaced by $(\Omega_\alpha, \Sigma_\alpha)$.

We can adapt the arguments in [19] and [10]. Under the assumptions in Proposition 5.5, if $\Lambda^{h,\Sigma}_{\mathbf{C}_1,\rho_1} = \Lambda^{h,\Sigma}_{\mathbf{C}_2,\rho_2}$, then the modulus in D_1 can be recovered, and as [10, Proposition 4.2], we have $\Lambda^{h,\Sigma_2}_{\mathbf{C}_1,\rho_1} = \Lambda^{h,\Sigma_2}_{\mathbf{C}_2,\rho_2}$. If the elastic modulus are piecewise analytic on each D_α, with interface Σ_α satisfying the same assumptions as Σ, we can continue this process and obtain the following theorem.

THEOREM 5.7. *Suppose that $\Lambda^{h,\Sigma}_{\mathbf{C}_1,\rho_1} = \Lambda^{h,\Sigma}_{\mathbf{C}_2,\rho_2}$. If on each subdomain D_α, $\mathbf{C}_j, \rho_j$, $j = 1, 2$ are analytic up to the boundary of D_α and satisfy either of the following conditions on D_α:*

1. *$\mathbf{C}_j$ is TI with a known symmetry axis;*
2. *Σ_α is (partly) flat, and $\mathbf{C}_j$ are orthorhombic with one of the three symmetry planes tangential to Σ_α for each α;*
3. *Σ_α is (partly) curved, and D_α, $\mathbf{C}_j, \rho_j$, $j = 1, 2$ are constant;*

then $\mathbf{C}_1 = \mathbf{C}_2$, $\rho_1 = \rho_2$.

Next we introduce the notion of subanalytic set: $A \subset \mathbb{R}^3$ is said to be subanalytic if for any $x \in \overline{A}$, there exists an open neighorhood U of x, real analytic compact manifolds $Y_{i,j}$, $i = 1, 2$, $1 \leq j \leq N$, and real analytic maps $\Phi_{i,j} : Y_{i,j} \to \mathbb{R}^3$ such that

$$A \cap U = \cup_{j=1}^N (\Phi_{1,j}(Y_{1,j}) \setminus \Phi_{2,j}(Y_{2,j})) \cap U.$$

For more details and nice properties about subanalytic sets, we refer the reader to [23] and [10]. As an example, we note here that a polyhedron with a piecewise analytic boundary is a subanalytic set. We also emphasize that the family of subanalytic sets is closed under finite union and finite intersection. Moreover, for two relatively compact subanlytic subsets A and B, the number of connected components of $A \cap B$ is always finite. With this property, if we have two domain partitioning $\overline{\Omega} = \cup_\alpha \overline{D^{(1)}_\alpha} = \cup_\beta \overline{D^{(2)}_\beta}$ by two sets of subdomains $D^{(1)}_\alpha$'s and $D^{(2)}_\beta$'s such that each set of subdomains are mutually disjoint subanalytic sets, we can consider the finer domain partitioning

$$\overline{\Omega} = \cup_\gamma \overline{\tilde{D}_\gamma},$$

where each $\tilde{D}_\gamma$ is a connected component of $D^{(1)}_\alpha \cap D^{(2)}_\beta$ for some α and β. Therefore, with the subanalytic property of the subdomains, we can recover the domain partitioning as well by adapting the argument of [10].

THEOREM 5.8. *Suppose that $\Lambda^{h,\Sigma}_{\mathbf{C}_1,\rho_1} = \Lambda^{h,\Sigma}_{\mathbf{C}_2,\rho_2}$ and Σ is curved. Let on each subdomain $D^{(j)}_\alpha$, $\mathbf{C}_j, \rho_j$ be constant for $j = 1, 2$. Let, furthermore, Ω and each $D^{(j)}_\alpha$ be open subanalytic subsets of $\mathbb{R}^3$, and all the boundaries $\partial D^{(j)}_\alpha \setminus \partial\Omega$ contain no open flat subsets. Then $\mathbf{C}_1 = \mathbf{C}_2$ and $\rho_1 = \rho_2$.*

Moreover, for isotropic elasticity, that is,

$$C^{ijkl} = \lambda(\delta^{ij}\delta^{kl}) + \mu(\delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk}), \tag{5.1}$$

we have the following corollary.

COROLLARY 5.9. *Suppose that $\Lambda^{h,\Sigma}_{\mathbf{C}_1,\rho_1} = \Lambda^{h,\Sigma}_{\mathbf{C}_2,\rho_2}$ and $\mathbf{C}$ is isotropic of the form (5.1). Let on each subdomain $D^{(j)}_\alpha$, λ_j, μ_j, ρ_j be analytic for $j = 1, 2$. Let, furthermore, Ω and each $D^{(j)}_\alpha$ be open subanalytic subsets of $\mathbb{R}^3$. Then $\lambda_1 = \lambda_2$, $\mu_1 = \mu_2$, and $\rho_1 = \rho_2$.*

The first and third authors, with collaborators, proved a uniqueness and Lipschitz stability for piecewise homogeneous isotropic elastic parameters λ, μ, ρ with a time-harmonic DN map [4]. From the uniqueness point of view, the above theorem is a more general result with nice enough properties of domain partitioning.

Appendix A. Reconstruction of density and stiffness tensor at the boundary from the surface impedance tensor. We present the reconstruction scheme for the material parameters (with certain symmetries) at the boundary from the surface impedance tensor Z introduced above.

A.1. Vertically transversely isotropic case. We first consider the vertically transversely isotropic case. We assume that Σ is flat and let the outer normal unit vector be $\mathbf{n} = (0,0,1)$ with respect to the Cartesian coordinates $y = (y^1, y^2, y^3)$; we assume that the axis of symmetry is aligned with this normal. Then the nonvanishing components of the VTI stiffness tensor, $\mathbf{C}$, are

$$C^{1111},\ C^{2222},\ C^{3333},\ C^{1122},\ C^{1133},\ C^{2233},\ C^{2323},\ C^{1313},\ C^{1212} \tag{A.1}$$

with relations

$$C^{1111} = C^{2222}, \quad C^{1133} = C^{2233},$$
$$C^{2323} = C^{1313}, \quad C^{1212} = \frac{1}{2}(C^{1111} - C^{1122}).$$

The strong convexity condition is equivalent to

$$C^{1313} > 0,\ C^{1212} > 0,\ C^{3333} > 0,\ (C^{1111} + C^{1122})C^{3333} > 2(C^{1133})^2.$$

In a neighborhood of Σ, we can use boundary normal coordinates x and the Cartesian coordinates y identically. We use the notation in section 4. With fixed $\mathbf{n}$, D, Q, R take the forms

$$D = \begin{pmatrix} C^{1313} & 0 & 0 \\ 0 & C^{1313} & 0 \\ 0 & 0 & C^{3333} \end{pmatrix}, \qquad R(\mathbf{m}) = \begin{pmatrix} 0 & 0 & C^{1133}m_1 \\ 0 & 0 & C^{1133}m_2 \\ C^{1313}m_1 & C^{1313}m_2 & 0 \end{pmatrix},$$
$$Q(\mathbf{m}) = \begin{pmatrix} C^{1111}m_1^2 + C^{1212}m_2^2 & (C^{1212} + C^{1122})m_1m_2 & 0 \\ (C^{1212} + C^{1122})m_1m_2 & C^{1212}m_1^2 + C^{1111}m_2^2 & 0 \\ 0 & 0 & C^{1313}(m_1^2 + m_2^2) \end{pmatrix}.$$

In above notations, the dependence on $\mathbf{n}$ is suppressed. Writing

$$P(\mathbf{m}) = \begin{pmatrix} |\mathbf{m}|^{-1}m_2 & |\mathbf{m}|^{-1}m_1 & 0 \\ -|\mathbf{m}|^{-1}m_1 & |\mathbf{m}|^{-1}m_2 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

we find that

$$\hat{D} = P(\mathbf{m})^* D P(\mathbf{m}) = \begin{pmatrix} C^{1313} & 0 & 0 \\ 0 & C^{1313} & 0 \\ 0 & 0 & C^{3333} \end{pmatrix},$$
$$\hat{R}(\mathbf{m}) = P(\mathbf{m})^* R(\mathbf{m}) P(\mathbf{m}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & C^{1133}|\mathbf{m}| \\ 0 & C^{1313}|\mathbf{m}| & 0 \end{pmatrix},$$
$$\hat{Q}(\mathbf{m}) = P(\mathbf{m})^* Q(\mathbf{m}) P(\mathbf{m}) = \begin{pmatrix} C^{1212}|\mathbf{m}|^2 & 0 & 0 \\ 0 & C^{1111}|\mathbf{m}|^2 & 0 \\ 0 & 0 & C^{1313}|\mathbf{m}|^2 \end{pmatrix}.$$

We emphasize the block-diagonal structure of the above matrices, and our later calculations will rely on this. We note that $P(\mathbf{m})$ acts as a block-diagonalizer of D, R, Q in the above calculation. Without this block-diagonalization, the calculation of $Z(\mathbf{m})$ would not be possible.

Remark A.1. We note that the block-diagonal structure is closely related to the decoupling of surface wave modes. The 1-by-1 block corresponds to Love waves, and the 2-by-2 block corresponds to Rayleigh waves. We refer the reader to [13] for further discussion.

Exploiting the commutativity, $DP(\mathbf{m}) = P(\mathbf{m})D$, we obtain the decomposition

$$S_0(\mathbf{m}) = P(\mathbf{m})D^{-1/2}(A + \mathrm{i}B)D^{1/2}P(\mathbf{m})^*,$$

where

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\alpha_1 \\ 0 & -\alpha_2 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} \tag{A.2}$$

with

$$\begin{aligned} \alpha_1 &= \frac{1}{1+\gamma}\frac{(C^{1133}+C^{1313})|\mathbf{m}|}{\sqrt{C^{1313}C^{3333}}}, \quad \alpha_2 = \gamma\alpha_1, \\ c &= \sqrt{\frac{C^{1313}|\mathbf{m}|^2+\rho}{C^{3333}} - \frac{(C^{1133}+C^{1313})^2|\mathbf{m}|^2}{(1+\gamma)^2C^{1313}C^{3333}}}, \quad b = \gamma c, \\ a &= \sqrt{\frac{C^{1212}|\mathbf{m}|^2+\rho}{C^{1313}}}, \quad \gamma = \sqrt{\frac{(C^{1111}|\mathbf{m}|^2+\rho)C^{3333}}{(C^{1313}|\mathbf{m}|^2+\rho)C^{1313}}}. \end{aligned}$$

Then

$$Z(\mathbf{m}) = -\mathrm{i}(DS_0 + R^T) = P(\mathbf{m}) \tag{A.3}$$
$$\begin{pmatrix} C^{1313}a & 0 & 0 \\ 0 & C^{1313}b & \mathrm{i}\sqrt{C^{1313}C^{3333}}\alpha_1 - \mathrm{i}C^{1313}|\mathbf{m}| \\ 0 & \mathrm{i}\sqrt{C^{1313}C^{3333}}\alpha_2 - \mathrm{i}C^{1133}|\mathbf{m}| & C^{3333}c \end{pmatrix} P(\mathbf{m})^*.$$

A.2. Orthorhombic case. We assume, as before, that Σ is flat and let the outer normal unit vector be $\mathbf{n} = (0, 0, 1)$ with respect to the Cartesian coordinates $y = (y^1, y^2, y^3)$; we assume that the coordinate axes span the symmetry planes. Then the nonvanishing components of $\mathbf{C}$ are

$$C^{1111},\ C^{2222},\ C^{3333},\ C^{1122},\ C^{1133},\ C^{2233},\ C^{2323},\ C^{1313},\ C^{1212}. \tag{A.4}$$

The matrices D, Q, R take the form

$$D = \begin{pmatrix} C^{1313} & 0 & 0 \\ 0 & C^{2323} & 0 \\ 0 & 0 & C^{3333} \end{pmatrix}, \quad R = \begin{pmatrix} 0 & 0 & C^{1133}m_1 \\ 0 & 0 & C^{2233}m_2 \\ C^{1313}m_1 & C^{2323}m_2 & 0 \end{pmatrix},$$
$$Q = \begin{pmatrix} C^{1111}m_1^2 + C^{1212}m_2^2 & (C^{1212}+C^{1122})m_1m_2 & 0 \\ (C^{1212}+C^{1122})m_1m_2 & C^{1212}m_1^2 + C^{2222}m_2^2 & 0 \\ 0 & 0 & C^{1313}m_1^2 + C^{2323}m_2^2 \end{pmatrix}.$$

We have block-diagonalizing matrix as for the VTI case. For particular directions of $\mathbf{m}$, we have the following block-diagonal structure. For $\mathbf{m} = (0, |\mathbf{m}|)$, we find that

$$R(|\mathbf{m}|\mathbf{e}_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & C^{2233}|\mathbf{m}| \\ 0 & C^{2323}|\mathbf{m}| & 0 \end{pmatrix},$$
$$Q(|\mathbf{m}|\mathbf{e}_2) = \begin{pmatrix} C^{1212}|\mathbf{m}|^2 & 0 & 0 \\ 0 & C^{2222}|\mathbf{m}|^2 & 0 \\ 0 & 0 & C^{2323}|\mathbf{m}|^2 \end{pmatrix}$$

so that

(A.5)
$$Z(|\mathbf{m}|\mathbf{e}_2) = \begin{pmatrix} C^{1313}a^1 & 0 & 0 \\ 0 & C^{2323}b^1 & \mathrm{i}\sqrt{C^{2323}C^{3333}}\alpha_1^1 - \mathrm{i}C^{2323}|\mathbf{m}| \\ 0 & \mathrm{i}\sqrt{C^{2323}C^{3333}}\alpha_2^1 - \mathrm{i}C^{2233}|\mathbf{m}| & C^{3333}c^1, \end{pmatrix},$$

where

$$\alpha_1^1 = \frac{1}{1+\gamma^1}\frac{(C^{2233}+C^{2323})|\mathbf{m}|}{\sqrt{C^{2323}C^{3333}}}, \quad \alpha_2^1 = \gamma^1\alpha_1^1,$$
$$c^1 = \sqrt{\frac{C^{2323}|\mathbf{m}|^2+\rho}{C^{3333}} - \frac{(C^{2233}+C^{2323})^2|\mathbf{m}|^2}{(1+\gamma^1)^2C^{2323}C^{3333}}}, \quad b^1 = \gamma^1 c^1,$$
$$a^1 = \sqrt{\frac{C^{1212}|\mathbf{m}|^2+\rho}{C^{1313}}}, \quad \gamma^1 = \sqrt{\frac{(C^{2222}|\mathbf{m}|^2+\rho)C^{3333}}{(C^{2323}|\mathbf{m}|^2+\rho)C^{2323}}}.$$

Similarly, for $\mathbf{m} = (|\mathbf{m}|, 0)$, we obtain

(A.6)
$$Z(|\mathbf{m}|\mathbf{e}_1) = \begin{pmatrix} C^{1313}b^2 & 0 & \mathrm{i}\sqrt{C^{1313}C^{3333}}\alpha_1^2 - \mathrm{i}C^{1313}|\mathbf{m}| \\ 0 & C^{2323}a^2 & 0 \\ \mathrm{i}\sqrt{C^{1313}C^{3333}}\alpha_2^2 - \mathrm{i}C^{1133}|\mathbf{m}| & 0 & C^{3333}c^2 \end{pmatrix}$$

with

$$\alpha_1^2 = \frac{1}{1+\gamma^2}\frac{(C^{1133}+C^{1313})|\mathbf{m}|}{\sqrt{C^{1313}C^{3333}}}, \quad \alpha_2^2 = \gamma^2\alpha_1^2,$$
$$c^2 = \sqrt{\frac{C^{1313}|\mathbf{m}|^2+\rho}{C^{3333}} - \frac{(C^{1133}+C^{1313})^2|\mathbf{m}|^2}{(1+\gamma^2)^2C^{1313}C^{3333}}}, \quad b^2 = \gamma^2 c^2,$$
$$a^2 = \sqrt{\frac{C^{1212}|\mathbf{m}|^2+\rho}{C^{2323}}}, \quad \gamma^2 = \sqrt{\frac{(C^{1111}|\mathbf{m}|^2+\rho)C^{3333}}{(C^{1313}|\mathbf{m}|^2+\rho)C^{1313}}}.$$

A.3. The reconstruction scheme. In this section, we give a reconstruction scheme for the material parameters. The VTI and orthorhombic cases have the same structure: We only need to consider the $Z^{(11)}(|\mathbf{m}|\mathbf{e}_1)$ element in (A.5) and the

$Z^{(11)}(|\mathbf{m}|\mathbf{e}_2)), Z^{(13)}(|\mathbf{m}|\mathbf{e}_2), Z^{(31)}(|\mathbf{m}|\mathbf{e}_2)$, and $Z^{(33)}(|\mathbf{m}|\mathbf{e}_2)$ elements in (A.6). Indeed, we only need to consider the VTI case.

We first make some basic algebraic observations. We note that Z is a Hermitian matrix and contains 4 nonzero elements for the VTI case. However, we have a total number of 6 unknowns to recover. This is feasible because these elements are nonhomogeneous in $\mathbf{m}$. Hence, different values for $|\mathbf{m}|$ give different information. It also becomes clear why the VTI or orthorhombic elastic parameters cannot be recovered from elastostatic data [32, 33]. In the VTI case, the surface impedance tensor for elastostatics is homogeneous in $\mathbf{m}$, and thus we can only have 4 equations for 5 parameters. Then it is impossible to recover all the parameters.

We begin with a basic lemma.

LEMMA A.2. *Consider a rational function*

$$f(t) = a + \frac{c}{t+b}$$

defined on $(0, \infty)$*; then* a, b, c *can be recovered from the values of* $f(1), f'(1), f''(1)$.

Proof. It is immediate that

$$f'(1) = -\frac{c}{(1+b)^2}, \qquad f''(1) = \frac{2c}{(1+b)^3},$$

from which we recover

$$b = \frac{2f'(1)}{f''(1)} - 1.$$

Then we recover

$$a = f(1) + f'(1)(1+b)$$

and then

$$c = (f(1) - a)(1+b),$$

completing the reconstruction. ∎

Step 1. From $(Z^{(11)}(|\mathbf{m}|\mathbf{e}_2)^2 = C^{1212}C^{1313}|\mathbf{m}|^2 + \rho C^{1313}$. By taking the difference with $|\mathbf{m}| = 1$ and $|\mathbf{m}| = \sqrt{2}$, we recover

$$C^{1212}C^{1313} = (Z^{(11)}(\sqrt{2}\mathbf{e}_2)^2 - (Z^{(11)}(\mathbf{e}_2))^2$$

and

$$\rho C^{1313} = (Z^{(11)}(\mathbf{e}_2))^2 - C^{1212}C^{1313}.$$

Step 2. From $Z^{(22)}(|\mathbf{m}|\mathbf{e}_2))$ and $Z^{(33)}(|\mathbf{m}|\mathbf{e}_2))$, we recover

$$d_1(|\mathbf{m}|^2) := \frac{C^{1313}}{C^{3333}} \frac{C^{1111}|\mathbf{m}|^2 + \rho}{C^{1313}|\mathbf{m}|^2 + \rho} = \frac{(Z^{(22)}(|\mathbf{m}|\mathbf{e}_2))^2}{(Z^{(33)}(|\mathbf{m}|\mathbf{e}_2))^2}.$$

Viewed as a rational function defined on $(0, \infty)$,

$$d(t) = d_1(t^2) = \frac{C^{1111}}{C^{3333}} + \frac{(\rho C^{1313} - \rho C^{1111})/(C^{1313}C^{3333})}{t + \frac{\rho}{C^{1313}}},$$

and applying Lemma A.2, from the values of $d(1), d'(1), d''(1)$ we recover

$$\frac{C^{1111}}{C^{3333}}, \qquad \frac{\rho}{C^{1313}} \quad \text{and} \quad \frac{\rho C^{1313} - \rho C^{1111}}{C^{1313}C^{3333}}.$$

The recovery of

$$\frac{\rho}{C^{3333}} = \frac{\rho C^{1313} - \rho C^{1111}}{C^{1313}C^{3333}} + \frac{\rho}{C^{1313}}\frac{C^{1111}}{C^{3333}}$$

follows immediately.

Step 3. With these recoveries, we successively obtain $\rho, C^{1313}, C^{1212}, C^{3333}$, and then C^{1111}.

Step 4. Now we have sufficient information to recover

$$\gamma = \sqrt{\frac{(C^{1111}|\mathbf{m}|^2 + \rho)C^{3333}}{(C^{1313}|\mathbf{m}|^2 + \rho)C^{1313}}}$$

as a function of $|\mathbf{m}|$. We note that

$$\frac{C^{1313}}{C^{3333}}(Z^{(33)}(|\mathbf{m}|\mathbf{e}_2))^2 = (C^{1313})^2|\mathbf{m}|^2 + \rho C^{1313} - \frac{(C^{1133} + C^{1313})^2|\mathbf{m}|^2}{(1+\gamma)^2}.$$

Thus, we recover

$$C^{1133} = (1+\gamma(1))\sqrt{-\frac{C^{1313}}{C^{3333}}(Z^{(33)}(\mathbf{e}_2))^2 + (C^{1313})^2 + \rho C^{1313}} - C^{1313}.$$

Step 5. We proceed with recovering the partial derivatives of the material parameters. Recall that the derivatives of Z can be recovered from the full symbols of DN map. From the the above procedure and Lemma A.2, we obtain

$$\partial\left(\frac{\rho}{C^{1313}}\right) = 2\partial\left(\frac{d'(1)}{d''(1)}\right)$$

and

$$\partial(\rho C^{1313}) = \partial\left(2(Z^{(11)}(\mathbf{e}_2))^2 - (Z^{(11)}(\sqrt{2}\mathbf{e}_2))^2\right)$$

where ∂ stands for any ∂_{y^j}, $j = 1,2,3$. It follows that

$$C^{1313}\partial\rho - \rho\partial C^{1313} = 2\partial\left(\frac{d'(1)}{d''(1)}\right)(C^{1313})^2,$$
$$C^{1313}\partial\rho + \rho\partial C^{1313} = \partial\left(2(Z^{(11)}(\mathbf{e}_2))^2 - (Z^{(11)}(\sqrt{2}\mathbf{e}_2))^2\right).$$

Solving the 2-by-2 linear system, we recover $\partial\rho$ and ∂C^{1313}.

Step 6. From the relation

$$C^{1212}\partial C^{1313} + C^{1313}\partial C^{1212} = \partial\left((Z^{(11)}(\sqrt{2}\mathbf{e}_2))^2 - (Z^{(11)}(\mathbf{e}_2))^2\right),$$

we recover

$$\partial C^{1212} = \frac{1}{C^{1313}}\left(C^{1212}\partial C^{1313} - \partial\left((Z^{(11)}(\sqrt{2}\mathbf{e}_2))^2 - (Z^{(11)}(\mathbf{e}_2))^2\right)\right).$$

Step 7. Again, following Lemma A.2,

$$\partial\left(\frac{C^{1111}}{C^{3333}}\right) = \partial\left(d(1) + d'(1)\left(1 + \frac{\rho}{C^{1313}}\right)\right).$$

Step 8. We note that

$$\partial\left(\frac{\rho C^{1313}-\rho C^{1111}}{C^{1313}C^{3333}}\right)=\partial\left(\left(d(1)-\frac{C^{1111}}{C^{3333}}\right)\left(1+\frac{\rho}{C^{1313}}\right)\right),$$

whence

$$C^{3333}\partial\rho-\rho\partial C^{3333}=(C^{3333})^2\partial\left(\left(d(1)-\frac{C^{1111}}{C^{3333}}\right)\left(1+\frac{\rho}{C^{1313}}\right)+\frac{\rho}{C^{1313}}\frac{C^{1111}}{C^{3333}}\right).$$

We recover ∂C^{3333}. Furthermore, we note that

$$C^{3333}\partial C^{1111}-C^{1111}\partial C^{3333}=(C^{3333})^2\partial\left(d(1)+d'(1)\left(1+\frac{\rho}{C^{1313}}\right)\right),$$

from which we recover ∂C^{1111}.

Step 9. We recover

$$\partial C^{1133}=\partial\left((1+\gamma(1))\sqrt{-\frac{C^{1313}}{C^{3333}}(Z^{(33)}(\mathbf{e}_2))^2+(C^{1313})^2+\rho C^{1313}}-C^{1313}\right).$$

Step 10. We recover higher-order derivatives

$$\partial^m\rho,\ \partial^m C^{1111},\ \partial^m C^{1313},\ \partial^m C^{3333},\ \partial^m C^{1133},\ \partial^m C^{1212}$$

from $\partial^m Z$ for $m=2,3,\dots$, where ∂^m stands for any ∂_y^α with $|\alpha|=m$. The procedures are similar to those for the recovery of the first-order derivatives.

Appendix B. Reconstruction of constant density and stiffness tensor and interface curvature condition. In this section, we assume that the density and stiffness tensor are constant. We revisit operator $\mathcal{M}$, defined in (1.3), and denote

$$M(\mathbf{m}+q\mathbf{n})=D(\mathbf{n})q^2+(R(\mathbf{m},\mathbf{n})+R(\mathbf{m},\mathbf{n})^T)q+Q(\mathbf{m})+\rho I,$$

where D,R,Q are all constant in y and $(\mathbf{m},q)$ is interpreted as the Fourier dual of y.

We develop the relation between the surface impedance tensor and the fundamental solution $\Gamma(y)$ of $\mathcal{M}$, satisfying

$$\mathcal{M}\Gamma(y)=\delta(y).$$

The semiclassical Fourier transform $\hat\Gamma(\eta)$ of Γ,

$$\hat\Gamma(\eta)=\int_{\mathbb{R}^3}e^{-\frac{\mathrm{i}y\cdot\eta}{h}}\Gamma(y)\mathrm{d}y,$$

satisfies

$$M(\mathbf{m}+q\mathbf{n})\hat\Gamma(\mathbf{m}+q\mathbf{n})=I.$$

In this expression,

$$\begin{aligned}M(\mathbf{m}+q\mathbf{n})&=(q-S_0(\mathbf{m},\mathbf{n})^*)D(q-S_0(\mathbf{m},\mathbf{n}))\\&=(q-\overline{S_0(\mathbf{m},\mathbf{n})}^*)D(q-\overline{S_0(\mathbf{m},\mathbf{n})}),\end{aligned}\tag{B.1}$$

where we use that $S_0^*D-DS_0=R+R^T=\overline{R+R^T}=\overline{S_0}^*D-D\overline{S_0}$, $S_0^*DS_0=Q+\rho I=\overline{S_0}^*T\overline{S_0}$. Here, again, $\mathrm{Spec}(S_0)\subset\mathbb{C}^+$, $\mathrm{Spec}(\overline{S_0})\subset\mathbb{C}^-$.

We denote the semiclassical inverse Fourier transform $\mathcal{F}^{-1}_{q\to\sigma,h}$ of $\hat\Gamma(\mathbf{m}+q\mathbf{n})$ by

$$\mathcal{F}^{-1}_{q\to\sigma,h}\hat\Gamma(\sigma,\mathbf{m},\mathbf{n})=\frac{1}{2\pi h}\int_{-\infty}^{+\infty}e^{\frac{\mathrm{i}q\sigma}{h}}\hat\Gamma(\mathbf{m}+q\mathbf{n})\mathrm{d}q.$$

Taking a semiclassical inverse Fourier transform in q of the identity

$$\begin{aligned}&\left(D(\mathbf{n})q^2+(R(\mathbf{m},\mathbf{n})+R(\mathbf{m},\mathbf{n})^T)q+Q(\mathbf{m})+\rho I\right)\hat\Gamma(\mathbf{m}+q\mathbf{n})\\&\quad=(q-S_0(\mathbf{m},\mathbf{n})^*)D(q-S_0(\mathbf{m},\mathbf{n}))\hat\Gamma(\mathbf{m}+q\mathbf{n})\\&\quad=I,\end{aligned}$$

we obtain

$$\begin{aligned}&\left(D(\mathbf{n})h^2D_\sigma^2+(R(\mathbf{m},\mathbf{n})+R(\mathbf{m},\mathbf{n})^T)hD_\sigma+Q(\mathbf{m})+\rho I\right)\mathcal{F}^{-1}_{q\to\sigma,h}\hat\Gamma(\sigma,\mathbf{m},\mathbf{n})\\&\quad=(hD_\sigma-S_0^*)D(hD_\sigma-S_0(\mathbf{m},\mathbf{n}))\mathcal{F}^{-1}_{q\to\sigma,h}\hat\Gamma(\sigma,\mathbf{m},\mathbf{n})\\&\quad=\delta(\sigma)I.\end{aligned}\tag{B.2}$$

Focus on the situation $\sigma>0$; considering the fact $\mathrm{Spec}(S_0^*)\subset\mathbb{C}^-$, we have

$$D(hD_\sigma-S_0)\mathcal{F}^{-1}_{q\to\sigma,h}\hat\Gamma(\sigma,\mathbf{m},\mathbf{n})=0\quad\text{for }\sigma>0.$$

Similarly, we have

$$D(hD_\sigma-\overline{S_0})\mathcal{F}^{-1}_{q\to\sigma,h}\hat\Gamma(\sigma,\mathbf{m},\mathbf{n})=0\quad\text{for }\sigma<0.$$

Integrating (B.2) in σ once and using the fact that $\mathcal{F}^{-1}_{q\to\sigma,h}\hat\Gamma(\sigma,\mathbf{m},\mathbf{n})$ is continuous in the variable σ, we have

$$\lim_{\sigma\to0+}hDD_\sigma\mathcal{F}^{-1}_{q\to\sigma,h}\hat\Gamma(\sigma,\mathbf{m},\mathbf{n})-\lim_{\sigma\to0-}hDD_\sigma\mathcal{F}^{-1}_{q\to\sigma,h}\hat\Gamma(\sigma,\mathbf{m},\mathbf{n})=I.$$

Then we conclude that

$$D(S_0-\overline{S_0})\mathcal{F}^{-1}_{q\to\sigma,h}\hat\Gamma(0,\mathbf{m},\mathbf{n})=I.$$

Recalling that

$$Z(\mathbf{m},\mathbf{n})=-\mathrm{i}(DS_0(\mathbf{m},\mathbf{n})+R^T(\mathbf{m},\mathbf{n})),$$

we find that

$$-\mathrm{i}D(S_0(\mathbf{m},\mathbf{n})-\overline{S_0}(\mathbf{m},\mathbf{n}))=2D\mathrm{Im}\{S_0(\mathbf{m},\mathbf{n})\}=2\mathrm{Re}\{Z(\mathbf{m},\mathbf{n})\}.$$

Therefore,

$$\mathcal{F}^{-1}_{q\to\sigma,h}\hat\Gamma(0,\mathbf{m},\mathbf{n})=\frac{1}{2}\left(\mathrm{Re}\{Z(\mathbf{m},\mathbf{n})\}\right)^{-1}.$$

We identify

$$X\hat\Gamma(\mathbf{m},\mathbf{n})=\int_{\mathbb{R}}\hat\Gamma(\mathbf{m}+s\mathbf{n})\mathrm{d}s=2\pi h\mathcal{F}^{-1}_{q\to\sigma,h}\hat\Gamma(0,\mathbf{m},\mathbf{n})\tag{B.3}$$

as the X-ray transform of $\hat{\Gamma}$ along $\mathbf{n}$. Thus,

$$\Gamma(y) = \int e^{-\frac{\mathrm{i}y\cdot\mathbf{m}}{h}} X\hat{\Gamma}(\mathbf{m},\mathbf{n})\mathrm{d}\mathbf{m}$$

for any $y \perp \mathbf{n}$. We say Σ is curved if Σ is locally represented by the graph of a function $y^3 = \varphi(y^1, y^2)$ such that $D^2\varphi$ does not vanish. If Σ is curved, we know $Z(\mathbf{m},\mathbf{n})$ for $\mathbf{n}$ in a continuous curve, joining two different points, on S^2. Then we know $\Gamma(y)$ for y in an open subset of $\mathbb{R}^3 \setminus \{0\}$. Since $\Gamma(y)$ is analytic in $\mathbb{R}^3 \setminus \{0\}$, we can recover $\Gamma(y)$, and thus $\hat{\Gamma}(\eta)$ for all $\eta \in \mathbb{R}^3$. Following [10] we then complete the reconstruction of the stiffness tensor $\mathbf{C}$ and the density ρ. The curved boundary condition is first introduced in [5] for the recovery of a piecewise homogeneous, fully anisotropic conductivity.

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