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Maarten V. de Hoop[a], Lauri Oksanen[b], and Justin Tittelfitz[c]

[a]Department of Computational and Applied Mathematics and Earth Science, Rice University, Houston, TX, USA; [b]Department of Mathematics, University College London, London, UK; [c]Department of Mathematics, Purdue University, West Lafayette, IN, USA

ABSTRACT

We consider an inverse source problem for an inhomogeneous wave equation with discrete-in-time sources, modeling a seismic rupture. The inverse source problem, with an arbitrary source term on the right-hand side of the wave equation, is not uniquely solvable. Here we formulate conditions on the source term that allow us to show uniqueness and that provide a reasonable model for the application of interest. We assume that the source term is supported on a finite set of times and that the support in space moves with subsonic velocity. Moreover, we assume that the spatial part of the source is singular on a hypersurface, an application being a seismic rupture along a fault plane. Given data collected over time on a detection surface that encloses the spatial projection of the support of the source, we show how to recover the times and locations of sources microlocally and then reconstruct the smooth part of the source assuming that it is the same at each source location.

1. Introduction

Let $c \in C^\infty(\mathbb{R}^n)$ be strictly positive and consider the wave equation

$$\begin{cases} \partial_t^2 u - c(x)^2 \Delta u = F(t, x) & \text{in } \mathbb{R} \times \mathbb{R}^n, \\ u(0, \cdot) = \partial_t u(0, \cdot) = 0 & \text{in } \mathbb{R}^n. \end{cases} \tag{1}$$

We will study the inverse source problem to determine F, given the data

$$\Lambda F := u|_{(0,T) \times \partial\Omega},$$

where $\Omega \subset \mathbb{R}^n$ is an open and bounded set with smooth boundary. It is well known that such a problem does not have a unique solution in general. For example, if we set $F = \partial_t^2 v - c^2 \Delta v$ where $v \in C_0^\infty(\Omega \times (0, T))$, then $\Lambda F = 0$.

To overcome nonuniqueness, we will assume that the source is of the form

$$F(t, x) := \sum_{j=1}^{J} \delta(t - t_j) f_j(x), \tag{2}$$

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where $J \in \mathbb{N}$ and $0 < t_1 < t_2 < \ldots < t_J$. Furthermore, we assume that f_j is in the space of compactly supported distributions $\mathcal{E}'(\Omega)$ and has the form

$$\langle f, \phi \rangle_{\mathcal{E}' \times C^\infty(\Omega)} = \int_{S_j} h_j(x)\phi(x)dx, \quad \phi \in C^\infty(\Omega), \tag{3}$$

where $S_j = \text{supp}(f_j)$ is a smooth oriented manifold with boundary and $h_j \in C^\infty(S_j)$. We assume that either $\dim(S_j) = n$ or $\dim(S_j) = n - 1$, and furthermore, that the extension of h_j by zero across ∂S_j is smooth in the case $\dim(S_j) = n - 1$, and that the extension is not smooth at any $x \in \partial S_j$ in the case $\dim(S_j) = n$.

We will reconstruct F in two steps. First we use a microlocal argument to recover the onset times t_j and supports S_j. Then we impose an assumption that the distributions f_j are translations of a single distribution f and that the translation speed is slower than the speed of wave propagation. The second step is the recovery of f. In the microlocal argument, we impose two generic assumptions that rule out certain cases that we consider degenerate, see (ML1) and (ML2) below.

Apart from the generic conditions (ML1) and (ML2), the above assumptions are motivated by models of seismic ruptures. The case $\dim(S_j) = n - 1$ is of particular interest, since rupture sources typically occur along a fault plane. It is also realistic to model a rupture using discrete-in-time sources, as the sources radiate strongly when the velocity of the rupture changes, which again happens typically during a short slip [42]. In the theory introduced by Madariaga [29], the radiation from a fault plane is controlled by the slip velocity in its ruptured portion. The slip velocity (and stress) has the property that it is strongly concentrated behind the rupture front. Barriers and asperities along the fault plane produce large variations of the intensities of these concentrations and are the source of high frequency waves. We refer to [2] for further discussion.

Ruptures propagate typically with a speed that is slower than the speed of wave propagation, and the assumption that the distributions f_j has the same spatial characteristics, although strong, is motivated by imaging results, see for example [46], where the radiated energies of the Denali and Kokoxili earthquakes are reconstructed using a back projection technique. Finally, let us point out that the assumption that $\partial\Omega$ encloses the supports S_j can be seen as an idealization of the fact that the ruptures happen inside the Earth and that the data are collected on its surface.

We mention the widely applied procedure for estimating the source by Kikuchi and Kanamori [22], which is based on maximizing the time correlations between observed and modeled wave solutions. Here, the ruptures are essentially represented by a sum of point sources parameterized by their locations and onset times. The sum of point source models a sequence of subevents in the rupture. A refined, iterative procedure introduces in every iteration a new subevent [28]. In our approach, we begin also by identifying the locations and onset times of subevents, however, in our case, the subevents have spatial structure modeled by f_j. The problem that we consider is called "kinematic inversion" in the seismic imaging literature.

1.1. Previous literature

Our proof uses the unique continuation principle by Tataru [41], see [36] and [18] for earlier results, and [6] and [11] for extensions to other time-dependent systems like elasticity. In

addition, we will draw upon ideas from the theory of inverse initial source problems, in particular, from [37] where a time-reversal approach for an inverse initial source problem with a nonconstant wave speed was introduced.

We emphasize that whereas the inverse source problem, with source on the right-hand side of the wave equation, is not uniquely solvable in general, the inverse initial source problem, source being the initial condition in this case, is always uniquely solvable. Let us also point out that even if it is assumed that there is only a single event, that is, if $J = 1$ in (2), the problem that we consider does not coincide with the inverse initial source problem. Indeed, to apply techniques from the theory of inverse initial source problems, the onset time t_1 needs to be recovered first.

To illustrate this further, let us assume for the moment that $J = 1$, $\dim(S_1) = n - 1$, and that the speed of wave propagation c is constant. Then the source is singular along the hypersurface S_1 and the two normal directions of S_1 generate two singular wave fronts that propagate in opposite directions along straight lines. In this case, our method finds the onset time t_1 by propagating the wave fronts backward from the measurement surface and by determining when they overlap. Once t_1 is known, any method that solves the inverse initial source problem, for example [37], can be used to recover the spatial structure f_1. In the general case, $J > 1$, information on the different events is mixed together in the measurement data, and this complicates the recovery of both the onset times and the spatial structure.

The motivation to study the inverse initial source problem in [37] was the medical imaging modality known as thermoacoustic tomography, but similar ideas have been used in many other applications, including geophysical ones. For time-reversal methods used in rupture detection, see [10, 21, 23, 24, 30, 35, 45], and in microseismicity, see [5, 9, 12]. Regarding the theory of inverse initial source problems, in addition to [37], see [47, 48] for the problem with partial data, see [38] for a speed with discontinuities, see [34] for numerical discussion, see [15, 43, 44] for the problem in elastic and attenuating media, respectively, and finally, one may find the surveys [19, 26, 27] of interest. There has also been recent work on the problem of jointly recovering the speed and source [39], and the problem of recovery with an approximate speed [31].

Let us now turn to inverse source problems, where the source is on the right-hand side of the wave equation. We mention the result by two of the authors [8], where a source of the form (2) is considered, but it is required that the sources are well separated from one another in space and time, in contrast to the subsonic proximity required in the current work. These assumptions are appropriate for modeling microseismicity (instead of ruptures, as in this paper). Most other results for inverse source problems consider a right-hand side of the form $a(t)f(x)$ or $a(t, x)f(x)$, where a is a known function, see [49] and [33], respectively, and [40] for a recent result. Similar problems have been stated and explored for the elastic wave equation, see [13] and [14].

2. Statement of the results

Before stating our results we need to introduce some notation. We begin by recalling the definition of the wave front set, see e.g., [16] for further details.

Definition 2.1. Let $X \subset \mathbb{R}^n$ be open. The *wavefront set* WF (w) of a distribution $w \in \mathcal{D}'(X)$ is a subset of the cotangent bundle T^*X indicating the locations and the directions of the

singularities of w. If $(x_0, \xi_0) \in T^*X \backslash 0$, then (x_0, ξ_0) is not in the wavefront set of w, if there exists $\psi \in C_0^\infty(X)$ with $\psi(x_0) \neq 0$, and a conic neighborhood V of ξ_0 such that

$$\left| \widehat{\psi w}(\xi) \right| \leq C_N (1 + |\xi|)^{-N}, \quad \xi \in V, \ N \in \mathbb{N}.$$

Here $\widehat{\psi w}$ indicates the Fourier transform of ψw.

If w satisfies the wave equation

$$\partial_t^2 w - c(x)^2 \Delta w = 0, \quad \text{in } \mathbb{R} \times \mathbb{R}^n,$$

then WF (w) is invariant under the bicharacteristic flow corresponding to the wave operator, see e.g., [17]. The principal symbol $p \in C^\infty(T^*\mathbb{R}^{1+n})$ of the wave operator is $p(t, x, \tau, \xi) = -\tau^2 + c^2(x)|\xi|^2$, and the forward bicharacteristic flow Φ acts on the level set $p^{-1}(0) \subset T^*\mathbb{R}^{1+n}$ as follows

$$\Phi : \mathbb{R} \times p^{-1}(0) \to p^{-1}(0), \quad \Phi(s; t, x, \tau, \xi) = (t + s\tau, \gamma(s), \tau, \dot{\gamma}(s)),$$

where $\gamma(s) = \gamma(s; x, \xi)$ is the geodesic on $(\mathbb{R}^n, c^{-2}dx^2)$ satisfying the initial conditions $\gamma(0) = x$ and $\dot{\gamma}(0) = \xi$. Here $\dot{\gamma}$ is the direction of γ as a cotangent vector, that is, in coordinates $\dot{\gamma} = c^{-2} \sum_{j=1}^n (\partial_s \gamma^j) dx^j$.

Let us now consider the solution u of (1) where F is of the form (2). For a set A, we denote by χ_A the indicator function of A, that is, $\chi_A(x) = 1$ if $x \in A$ and $\chi_A(x) = 0$ otherwise. By Duhamel's principle, it holds that $u = \sum_{j=1}^J u_j$ where $u_j = \chi_{\{t \geq t_j\}} w_j$ and w_j is the solution of

$$\begin{cases} \partial_t^2 w - c^2 \Delta w = 0, & \text{in } \mathbb{R} \times \Omega, \\ w(t_j, x) = 0; \quad \partial_t w(t_j, x) = f_j(x) & \text{in } \Omega. \end{cases} \tag{4}$$

Note that WF (u_j) is not invariant under the bicharacteristic flow Φ but WF (w_j) is.

We will next formulate three assumptions in terms of microlocal properties of the distributions f_j. We define the $n - 1$ dimensional manifold without boundary

$$\Sigma_j = \begin{cases} \partial S_j, & \dim(S_j) = n, \\ S_j^{\text{int}}, & \dim(S_j) = n - 1, \end{cases} \quad j = 1, \ldots, J, \tag{5}$$

and assume that

$$\text{WF } (f_j) = \overline{N^* \Sigma_j}, \quad j = 1, 2, \ldots, J. \tag{CN}$$

Here $N^* \Sigma_j$ is the conormal bundle of Σ_j. In the case $\dim(S_j) = n - 1$, we let ν to be one of the two unit conormal vector fields of Σ_j, and in the case $\dim(S_j) = n$, we let ν to be the outward unit outward conormal vector field of Σ_j. Then $\overline{N^* \Sigma_j}$ is the union of the following two sets

$$N_j^\pm = \{(x, a\nu) \in T^*\mathbb{R}^n; \ x \in \overline{\Sigma_j}, \ \pm a > 0\}.$$

Note that if $\dim(S_j) = n - 1$, then (CN) amounts to assuming that the extension of h_j in (3) by zero across ∂S_j is smooth. This follows from [16, Th. 8.1.5] together with a change of coordinates. In the case $\dim(S_j) = n$, (CN) means that the extension of h_j as above is not smooth at any $x \in \partial S_j$.

For each $(x, \xi) \in N_j^\pm$ there is unique $\tau > 0$ such that $(t_j, x, \tau, \xi) \in p^{-1}(0)$, and we write $P_j(x, \xi) = (t_j, x, \tau, \xi)$. Our second assumption is that the images of WF (f_j) are disjoint under

the bicharacteristic flow Φ in the sense that

$$\Phi(\mathbb{R} \times P_j(N_j^+ \cup N_j^-)) \cap \Phi(\mathbb{R} \times P_k(N_k^+ \cup N_k^-)) = \emptyset, \quad j \neq k. \tag{ML1}$$

This is equivalent to saying that there are no two points lying on different sets Σ_j such that the corresponding normal directions are tangent to the same geodesic on $(\mathbb{R}^n, c^{-2}dx^2)$. Furthermore, in terms of the solutions w_j of the problems (4), the condition (ML1) can be written briefly as

$$\mathrm{WF}\left(w_j\right) \cap \mathrm{WF}\left(w_k\right) = \emptyset, \quad j \neq k. \tag{ML1'}$$

As $\mathrm{WF}\left(w_j\right)$ is invariant under the bicharacteristic flow, it holds that $\mathrm{WF}\left(w_j(t, \cdot)\right)$ is the union of the two sets

$$\mathrm{WF}^{\pm}\left(w_j(t, \cdot)\right) = \{(\gamma(s), \dot{\gamma}(s)); \ s = t - t_j, \ \gamma = \gamma(\cdot; x, \xi), \ (x, \xi) \in N_j^{\pm}\}.$$

We call $\mathrm{WF}^+\left(w_j(t, \cdot)\right)$ and $\mathrm{WF}^-\left(w_j(t, \cdot)\right)$ the outward and inward wavefronts, respectively. Note that the outward and inward wavefronts pair at the source time t_j in the following sense

$$\mathrm{WF}^+\left(w_j(t_j, \cdot)\right) = N_j^+ = \widetilde{N_j^-} = \widetilde{\mathrm{WF}^-}\left(w_j(t_j, \cdot)\right), \tag{6}$$

where tilde indicates reflection in the dual variable, that is,

$$\widetilde{A} = \{(x, -\xi) : (x, \xi) \in A\}, \quad A \subset T^*\Omega. \tag{7}$$

Now we state our third microlocal assumption, that (6) is the only kind of pairing. That is, we assume that the manifolds Σ_j are connected and

$$\text{if } \mathrm{WF}^{\sigma}\left(w_j(t, \cdot)\right) = \widetilde{\mathrm{WF}^{\sigma'}}\left(w_k(t, \cdot)\right) \text{ then } j = k, \ t = t_j \text{ and } \sigma \neq \sigma'. \tag{ML2}$$

We denote by $S^*\Omega$ the unit cosphere bundle

$$S^*\Omega = \{(x, \xi) \in T^*\Omega; \ c^2(x)|\xi|^2 = 1\}.$$

We will prove the following theorem in Section 3.

Theorem 2.2. *Suppose that Ω is nontrapping and strictly convex in the sense that for all $(x, \xi) \in S^*\Omega$, the geodesic $\gamma = \gamma(\cdot; x, \xi)$ satisfies the following: there is unique $s \in (0, T - t_J)$ such that $\gamma(s) \in \partial\Omega$, and, furthermore, $\dot{\gamma}(s) \notin T^*_{\gamma(s)}\partial\Omega$ and $\gamma(t) \in \mathbb{R}^n \backslash \overline{\Omega}$ for all $t > s$. Suppose that the manifolds Σ_j are smooth and connected, and that (CN), (ML1) and (ML2) are satisfied. Then the times t_j and supports $S_j, j = 1, 2, \ldots, J,$ can be recovered from the boundary data ΛF.*

Let us give an example that does not satisfy (ML2). Let $n = 2$ and let S_1 and S_2 be two identical discs, so that the initial singular supports (that is, the projections of $\mathrm{WF}\left(f_j\right), j = 1, 2,$ to the base space Ω) are circles. Suppose that the wave speed c is constant. Then at $\frac{1}{2}(t_2 - t_1)$ the outgoing wavefront from the first source and the inward wavefront from the second source pair to form a larger circle, see Figure 1. Note, however, that if the spatial location of either of these discs is perturbed slightly, this pairing no longer occurs. In fact, we show in Section 7 that both the conditions (ML1) and (ML2) are generic.

We will next consider the case that the distributions f_j are obtained from a single distribution f through translations and show how to recover f. We will study two translation models: a
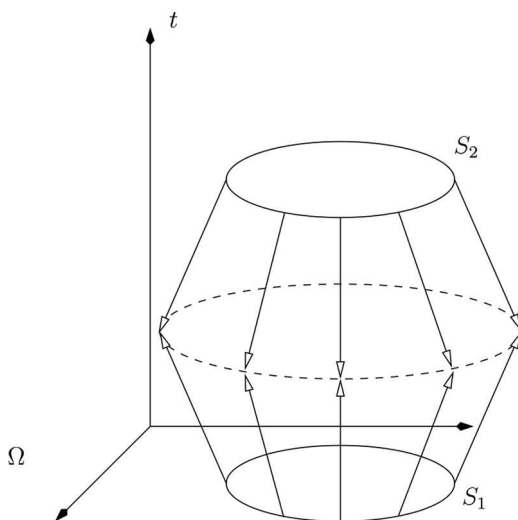
Figure 1. Two wavefronts forming a spurious pairing.

Riemannian parallel transport and the Euclidean translation. To simplify the notation, we assume that Ω contains the origin of $\mathbb{R}^n$.

Let us consider the Riemannian case first. We assume that the Riemannian manifold $(\Omega, c^{-2}dx^2)$ is simple, that is, Ω is simply connected, $\partial\Omega$ is strictly convex in the sense of the second fundamental form, and there are no conjugate points on Ω. We denote by

$$\mathcal{T}_x : T_0\Omega \to T_x\Omega, \quad x \in \Omega,$$

the parallel transport along the radial unit speed geodesic from the origin to x. That is, for each x, we choose $\xi \in S_0^*\Omega$ and $r \geq 0$ such that $x = \gamma(r; 0, \xi)$, and for each vector $v \in T_0\Omega$, we solve the equation

$$D_s V = 0, \quad V(0) = v,$$

where D_s is the covariant derivative of the metric $c^{-2}dx^2$ along the curve $\gamma(s; 0, \xi)$. Finally we set $\mathcal{T}_x v = V(r)$.

We assume that for each $j = 1, 2, \ldots, J$, there is $x_j \in \Omega$ such that

$$f_j = f \circ \mathcal{T}_{x_j}^{-1} \circ \exp_{x_j}^{-1}, \tag{R1}$$

where $\exp$ is the exponential map of $(\Omega, c^{-2}dx^2)$, $f \in \mathcal{E}'(T_0\Omega)$, and the precomposition means the pullback of f by $\mathcal{T}_{x_j}^{-1} \circ \exp_{x_j}^{-1}$, see e.g., [16, Th. 6.1.2] for the definition. Note that if $c = 1$ identically, then in coordinates, $\mathcal{T}_x$ is the identity and

$$f(v) = f_j \circ \exp_{x_j}(v) = f_j(x_j + v).$$

Thus f_j is obtained from f by an Euclidean translation.

We assume that

$$d(x_{j+1}, x_j) < t_{j+1} - t_j, \quad j = 1, 2, \ldots, J - 1, \tag{SS}$$

where $d(\cdot, \cdot)$ is the Riemannian distance function of $(\mathbb{R}^n, c^{-2}dx^2)$. Note that $d(x, y)$ gives the travel time distance between points $x, y \in \mathbb{R}^n$. We think of (SS) as a condition limiting the

speed at which the sources can propagate, effectively requiring this motion to be "subsonic," i.e., slower than the speed of wave propagation. Let us emphasize that the translation model (R1) considers only spatial variables and says nothing about the speed of the translation in space–time whereas (SS) requires that the speed is subsonic.

We will prove the following theorem in Section 4.3.

Theorem 2.3. *Suppose that the Riemannian manifold $(\Omega, c^{-2}dx^2)$ is simple and that (SS) and (R1) are satisfied. Suppose furthermore that the times t_j and the points x_j, $j = 1, 2, \ldots, J$, are known. If*

$$T > t_1 + \operatorname{diam}(\Omega), \tag{8}$$

where $\operatorname{diam}(\Omega) = \sup_{x,y \in \Omega} d(x,y)$, *then F can be recovered from the boundary data ΛF.*

To combine Theorems 2.2 and 2.3, we need to be able to determine the points x_j given the supports S_j. We will consider this problem in Section 5.

Let us now describe the Euclidean translation model. We assume that for each $j = 1, 2, \ldots, J$, there is $x_j \in \Omega$ such that

$$f_j(x) = f(x - x_j), \tag{E1}$$

where $f \in \mathcal{E}'(\Omega)$. Furthermore, we assume that in addition to the subsonic condition (SS), the following separation condition holds:

$$t_2 - t_1 > \frac{1 - c^-/c^+}{1 - \rho} R \tag{E2}$$

where $c^+ = \sup_{x \in \Omega} c(x)$, $c^- = \inf_{x \in \Omega} c(x)$ and

$$\rho = \max_{j=1,\ldots,J-1} \frac{d(x_{j+1}, x_j)}{t_{j+1} - t_j}, \quad R = \max_{j=1,\ldots,J} \min\{r > 0;\ S_j \subset B_r(x_j)\}. \tag{9}$$

Here $B_r(x)$ is the closed geodesic ball $\{y \in \mathbb{R}^n;\ d(y, x) \leq r\}$. Note that (SS) implies that $\rho \in [0, 1)$. We will prove the following theorem in Section 4.3.

Theorem 2.4. *Suppose that the Riemannian manifold $(\Omega, c^{-2}dx^2)$ is simple and that (SS), (E1) and (E2) are satisfied. Suppose furthermore that the times t_j and the points x_j, $j = 1, 2, \ldots, J$ are known. If T satisfies (8), then F can be recovered from the boundary data ΛF.*

If c is constant, then (R1) and (E1) are equivalent and (E2) is trivially satisfied. Without loss of generality, we may assume that f is defined, so that the center of mass of its support is at the origin. Then x_j is the center of mass of S_j and therefore S_j determines x_j, see Section 5 for more details. We will give further examples in Section 6.

Let us formulate one more result where, instead of a translation assumption as above, we assume the following strong separation condition: for some points x_j, suppose

$$(1 - \rho)(t_j - t_{j-1}) > 2R, \tag{TS}$$

where ρ and R are as in (9). This condition not only limits the speed at which the source can move, but it also implies a minimum gap in time between sources (of size roughly $2R$). This condition is stronger than (E2), but has the advantage of allowing completely distinct f_j and

arbitrary geometry $(\Omega, c^{-2}dx^2)$. Further, the condition depends not only on the S_j's and t_j's but also on the particular choice of x_j as well; the S_js do not enforce a natural choice for x_j as in the other scenarios, and if they are chosen poorly, then the resulting condition (TS) may not be optimal (relative to fixed collections of S_j and t_j). We prove the following theorem in Section 4.3.

Theorem 2.5. *Suppose that the conditions* (SS) *and* (TS) *are satisfied, and that the times t_j and the supports $S_j, j = 1, \ldots, J$ are known. If $T > t_J + \mathrm{diam}\,(\Omega)$, then F can be recovered from the boundary data ΛF.*

3. Microlocal identification

In this section, we prove Theorem 2.2. We define the exit time

$$\sigma_\Omega(x, \xi) = \max\left\{t \geq 0 : \gamma(t; x, \xi) \in \overline{\Omega}\right\}, \quad (x, \xi) \in T^*\mathbb{R}^n \backslash 0, \, x \in \overline{\Omega}.$$

Let $t \in \mathbb{R}$ and consider the map

$$\Psi_t : T^*\Omega \backslash 0 \to T^*(\mathbb{R} \times \partial\Omega), \quad \Psi_t(x, \xi) = (t + \sigma\tau, \gamma(\sigma), \tau, \dot{\gamma}'(\sigma)),$$

where $\tau = c(x)|\xi|$, $\sigma = \sigma_\Omega(x, \xi)$, $\gamma = \gamma(\cdot; x, \xi)$, and $\dot{\gamma}'$ is the projection of $\dot{\gamma}$ on $T^*\partial\Omega$. Note that Ψ_t is the composition of the restriction on $\{t\} \times \Omega$, the bicharacteristic flow Φ, and the restriction on $T^*(\mathbb{R} \times \partial\Omega)$. It is well known that Ψ_t is a local diffeomorphism if Ω is nontrapping and strictly convex. For the convenience of the reader, we give a proof here.

Lemma 3.1. *Suppose that Ω is nontrapping and strictly convex as formulated in Theorem 2.2. Let $t \in \mathbb{R}$. Then Ψ_t is an injective local diffeomorphism.*

Proof. We begin by showing that σ_Ω is smooth on $S^*\Omega$. Let $(x_0, \xi^0) \in T^*\Omega \backslash 0$. By the nontrapping assumption, $s_0 := \sigma_\Omega(x_0, \xi^0)$ is well defined and by the convexity assumption, $\dot{\gamma}(s_0; x_0, \xi^0)$ is not tangential to $\partial\Omega$. It follows from the implicit function theorem that the equation $\gamma(s; x, \xi) \in \partial\Omega$ has a unique solution s near s_0 that depends smoothly on (x, ξ) near (x_0, ξ^0). By the convexity assumption, this solution coincides with σ_Ω near (x_0, ξ^0). This shows that σ_Ω is smooth and therefore Ψ_t is smooth.

We will use boundary normal coordinates $y := (y^1, y') \in (-\epsilon, \epsilon) \times \partial\Omega$, where $\epsilon > 0$ is small. In these coordinates, the metric tensor $g := c^{-2}dx^2$ has the form

$$g(y) = \begin{pmatrix} 1 & 0 \\ 0 & h(y) \end{pmatrix}.$$

We denote by $|\eta|_g$ the norm of a cotangent vector $\eta = (\eta_1, \eta')$ with respect to the metric g and have $|\eta|_g^2 = \eta_1^2 + |\eta'|_h^2$.

We show next that Ψ_t is an immersion. Let $(x_0, \xi^0) \in T^*\Omega \backslash 0$ and define s_0 as above. We denote $\phi(x, \xi) = \gamma(s_0; x, \xi)$ and $\psi(x, \xi) = \dot{\gamma}(s_0; x, \xi)$. Let $p \in T_{(x_0, \xi^0)} T^*\Omega$ satisfy $d\Psi_t p = 0$. The third component of this equation says that $d\tau p = 0$ and therefore the first component implies that $d\sigma p = 0$. Now the second and fourth components imply $d\phi p = 0$ and $d\psi' p = 0$. Here we are using the notation $\psi = (\psi_1, \psi') \in \mathbb{R} \times \mathbb{R}^{n-1}$. As the geodesic flow is a diffeomorphism on $T^*\mathbb{R}^n$, $d\phi p = 0$ and $d\psi p = 0$ imply that $p = 0$. Thus it is enough to

show that $d\psi_1 p = 0$. As the geodesic flow preserves the norm, we have

$$0 = d\tau p = d|\psi|_g^2 p = 2\psi_1 d\psi_1 p + d|\psi'|_h^2 p.$$

As $\dot{\gamma}(s_0; x_0, \xi^0)$ is not tangential to $\partial\Omega$, we have $\psi_1 \neq 0$. Moreover,

$$d|\psi'|_h^2 p = 2\psi_j h^{jk} d\psi_k p + \psi_j dh^{jk} d\phi p \psi_k = 0,$$

whence $d\psi_1 p = 0$ and we have shown that Ψ_t is an immersion. As $T^*\Omega\backslash 0$ and $T^*(\mathbb{R} \times \partial\Omega)$ have the same dimension, Ψ_t is a local diffeomorphism.

It remains to show that Ψ_t is injective. Suppose that $(r, y, \tau, \eta') \in T^*(\mathbb{R} \times \partial\Omega)$ and that there is $(x, \xi) \in T^*\Omega\backslash 0$ such that $\Psi_t(x, \xi) = (r, y, \tau, \eta')$. Then $|\eta'|_g \leq \tau$ and there is a unique $a \geq 0$ such that $|\eta' + a\nu|_g = \tau$, where ν is the outward unit normal covector of $\partial\Omega$. By the convexity assumption, γ does not return to $\overline{\Omega}$ after σ, whence $\dot{\gamma}(\sigma) = \eta' + a\nu$. We have $\sigma = (r - t)/\tau$ and $(x, \xi) = (\beta(\sigma), \dot{\beta}(\sigma))$ where $\beta = \gamma(\cdot; y, -\eta' - a\nu)$. □

Proof of Theorem 2.2. We recall that

$$\Lambda F = u|_{(0,T)\times\partial\Omega} = \sum_{j=1}^{J} \chi_{\{t \geq t_j\}} w_j|_{(0,T)\times\partial\Omega}.$$

The map $f_j \mapsto w_j|_{(t_j,T)\times\partial\Omega}$ is a sum of two elliptic Fourier integral operators with canonical relations given by the graphs of Ψ_{t_j} and the composition of the reflection (7) and Ψ_{t_j}, respectively, see e.g., [37, Prop. 3]. As WF (f_j) is symmetric with respect to the reflection (7), we consider only Ψ_{t_j}. The assumption that unit speed geodesics exit Ω before time $T - t_J$ together with (ML1) implies that

$$\text{WF}(\Lambda F) = \bigcup_{j=1}^{J} \Psi_{t_j}\left(\text{WF}(f_j)\right).$$

By Lemma 3.1, the map Ψ_t is continuous and therefore it maps the connected components $\text{WF}^{\pm}(w_j(t, \cdot))$ of WF $(w_j(t, \cdot))$ to connected components of $\Psi_t(\text{WF}(w_j))$ assuming that WF $(w_j(t, \cdot)) \subset T^*\Omega$. Let us consider two connected components Γ_1 and Γ_2 of WF (ΛF) and let $t \in (t_0, t_1)$ where $t_0 \in \mathbb{R}$ is chosen to be the smallest possible time, so that $\Psi_t^{-1}(\Gamma_1 \cup \Gamma_2)$ is well defined (that is, the image stays in $T^*\Omega$) and

$$t_1 = \min\{r \in \mathbb{R}; \text{ there are } (y, \eta') \in T^*\partial\Omega \text{ and } \tau \in \mathbb{R} \text{ such that}$$
$$(r, y, \tau, \eta') \in \Gamma_1 \cup \Gamma_2\}.$$

Then $\Psi_t^{-1}(\Gamma_p) = \text{WF}^{\sigma_p}(w_{j_p}(t, \cdot))$, $p = 1, 2$, for some $\sigma_p = \pm$ and $j_p = 1, \ldots, J$. By (ML2), the sets $\Psi_t^{-1}(\Gamma_p)$, $p = 1, 2$, pair under the reflection (7) if and only if $j_1 = j_2$, $t = t_{j_1}$ and they coincide with the sets $N_{j_1}^{\pm}$.

The assumption (ML1) implies that there is a bijection between the connected components of WF (ΛF) and the sets $N_j^{\pm}$, $j = 1, \ldots, J$. Thus we can determine the times t_j and the sets $N_j^{\pm}$, $j = 1, \ldots, J$. □

We get the following partial data result by inspecting the proof of Theorem 2.2:

Remark 3.2. Consider the case where we know only a restriction of ΛF, that is, we know $u|_{(0,T)\times\omega}$ where $\omega \subset \partial\Omega$ is open. Then we can still recover the source times $t_j, j = 1, 2, \ldots, J$,

assuming a stronger form of (ML2). That is, the connected components Γ_k, $k = 1, 2, \ldots, K$, of WF $\left(u|_{(0,T)\times\omega}\right)$ are assumed to form pairs exactly at times t_j in the sense that if

$$\Psi_t^{-1}(\Gamma_{k_1}) \cap \widetilde{\Psi_t^{-1}(\Gamma_{k_2})} \neq \emptyset \tag{10}$$

then $t = t_j$ for some j, and that for all j, there are k_1 and k_2 such that (10) holds with $t = t_j$.

The condition in Remark 3.2 means first that ω needs to be large enough, so that we catch parts of all outward and inward wavefronts and that the outward and inward parts coming from the same source do not miss each other completely when propagated back using Ψ_t^{-1}, and second, that there are no spurious pairings.

Note that if $\Psi_t^{-1}(\Gamma_{k_1}) \subset \mathrm{WF}^+\left(w_j(t, \cdot)\right)$ and $\Psi_t^{-1}(\Gamma_{k_2}) \subset \mathrm{WF}^-\left(w_j(t, \cdot)\right)$, then the projection of the intersection (10) on the base space Ω is a subset of Σ_j, assuming that there are no spurious pairings. We can reconstruct this subset, but typically we can not reconstruct the whole set Σ_j from the partial data using the above microlocal argument. We will further discuss the partial data case in Remark 4.7 below.

Remark 3.3. In a procedure introduced by Ishii *et al.* [20] and quite commonly applied in seismology, the wavefield observed in (an open subset of) the boundary is reverse time continued and then restricted to a subset of a chosen hypersurface, $\Sigma \subset \Omega$ say, yielding $\sum_{j=1}^J w_j|_\Sigma$ without determining the t_i explicitly. As a matter of fact, this is done microlocally and referred to as back-projection with stacking (over the point receivers in the mentioned subset of the boundary). In the case $\dim S_i = n$, we can extend this procedure using our model as follows: If $S_i \cap \Sigma \neq \emptyset$ and there are no spurious pairings, then the paired components of the wavefront set of $\sum_{j=1}^J w_j|_{\Sigma, t=t_i}$ correspond to the two components of the conormal bundle of $S_i \cap \Sigma$ in $T^*\Sigma$, and this pairing can be recovered by our method.

4. Reconstruction of the smooth part of the source

4.1. Distances to geodesic balls

We begin by establishing two lemmas. Here (M, g) is a smooth compact Riemannian manifold with boundary. We define

$$\sigma_p(\xi) = \sup\{t > 0;\ \exp_p(t\xi) \in M^{\mathrm{int}}\}, \quad p \in M^{\mathrm{int}},\ \xi \in S_p M,$$

where SM denotes the unit sphere bundle of M, and $B_r(p) = \{x \in M;\ d(x, p) \leq r\}$, $r > 0$, where d denotes the distance function of M.

Lemma 4.1. *Suppose that ∂M is strictly convex in the sense of the second fundamental form. Let $p \in M^{\mathrm{int}}$ and let $R > 0$. Suppose that $S := B_R(p) \subset M^{\mathrm{int}}$, and that ∂S is smooth. Let $y \in \partial M$ and suppose that $x \in S$ satisfies*

$$d(y, x) = d(y, S). \tag{11}$$

Then there is $\xi \in S_p M$ such that

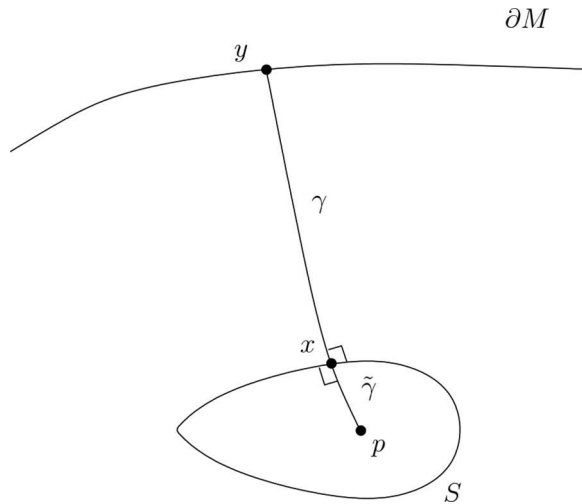$$x = \exp_p(R\xi) \quad and \quad y = \exp_p(\sigma_p(\xi)\xi). \tag{12}$$

Figure 2. The geodesics γ and $\widetilde{\gamma}$ coincide.

Proof. Clearly $x \in \partial S$ and there is $\xi \in S_p M$ such that $x = \exp_p(R\xi)$. Let $\gamma : [0, \ell] \to M$ be a shortest path from x to y. Then γ is C^1 and we may assume without loss of generality that it has unit speed [4].

A shortcut argument shows that $\dot{\gamma}(0) \perp \partial S$. Thus $\gamma(t)$ coincides with the path $\widetilde{\gamma}(t) = \exp_p((t + R)\xi)$ until it hits the boundary ∂M at $t = \sigma_p(\xi) - R$ (Figure 2). As ∂M is strictly convex, $\dot{\gamma}(\sigma_p(\xi) - R)$ is not tangential to the boundary ∂M. This implies that $\sigma_p(\xi) - R = \ell$, since otherwise γ cannot be C^1. $\qquad\square$

In general, there might exist $x \in \partial S$ such that

$$d(y, x) > d(y, S), \quad \text{for all } y \in \partial M.$$

However, in the case of a simple manifold, this cannot happen.

Lemma 4.2. *Suppose that (M, g) is simple. Let $p \in M^{int}$ and let $R > 0$. Suppose that $S := B_R(p) \subset M^{int}$. Let $\xi \in S_p M$ and define $x \in \partial S$ and $y \in \partial M$ by* (12). *Then* (11) *holds.*

Proof. Note that $\partial B_R(p)$ is smooth. As S is compact, there is a point $z \in \partial S$ such that $d(y, z) = d(y, S)$. Lemma 4.1 implies that there is $\zeta \in S_p M$ such that

$$z = \exp_p(R\zeta) \quad \text{and} \quad y = \exp_p(\sigma_p(\zeta)\zeta).$$

The map $\exp_p$ is injective by the simplicity, whence $\zeta = \xi$. In particular, $z = x$ and (11) holds. $\qquad\square$

4.2. Unique continuation

The following time-sharp semi-global unique continuation result follows from the seminal local result by Tataru [41].

Theorem 4.3. *Let $h \in C(\partial\Omega)$ and define*

$$\Gamma(h) = \{(t, y) \in \mathbb{R} \times \partial\Omega; \ |t| < h(y)\}, \quad T = \max_{y \in \partial\Omega} h(y).$$

Let $s \in \mathbb{R}$, and suppose that $w \in H^s((-T, T) \times \mathbb{R}^n)$ satisfies $\partial_t^2 w - c^2 \Delta w = 0$ and

$$w|_{\Gamma(h)} = 0, \quad \partial_\nu w|_{\Gamma(h)} = 0. \tag{13}$$

Then $w = 0$ and $\partial_t w = 0$ on $\{0\} \times \Omega(h)^{int}$, where

$$\Omega(h) = \{x \in \Omega; \ \text{there is } y \in \partial\Omega \text{ such that } d(x, y) \leq h(y)\}.$$

Proof. See [25, Th. 3.16] for a proof in the case that $s = 1$ and that $\Gamma(h)$ is replaced by a cylinder the form $(-R, R) \times \Gamma$, where $R > 0$ and $\Gamma \subset \partial\Omega$ are open. We will reduce the general case to this case by approximating $\Gamma(h)$ with a union of cylinders and by approximating w with a smooth function. Note that all the four traces of w in the formulation of the theorem are well defined in the sense of [16, Corollary 8.2.7] since WF (w) is a subset of the characteristic set $p^{-1}(0)$.

We begin by considering the case $s = 1$. Let $x \in \Omega(h)^{\text{int}}$. Then there is $y \in \partial\Omega$ such that $d(x, y) < h(y)$, and whence there exist a neighbourhood $\Gamma \subset \partial\Omega$ of y and $R > d(x, y)$ such that $(-R, R) \times \Gamma \subset \Gamma(h)$. Now [25, Th. 3.16] implies that w vanishes in a neighborhood of $(0, x)$. As $x \in \Omega(h)^{\text{int}}$ was arbitrary, we see that $w = 0$ and $\partial_t w = 0$ on $\{0\} \times \Omega(h)^{\text{int}}$.

Let us now show that the case of arbitrary $s \in \mathbb{R}$ can be reduced to the case $s = 1$. Let $\epsilon > 0$, $\psi \in C_0^\infty(-\epsilon, \epsilon)$, let us extend w by zero to $\mathbb{R} \times \mathbb{R}^n$ while denoting the extension still by w, and let $\tilde{w}$ be the convolution in the time variable $\tilde{w} = \psi * w$. As the operator $\partial_t^2 - c^2 \Delta$ commutes with the map $w \mapsto \psi * w$, the distribution $\tilde{w}$ satisfies

$$\partial_t^2 \tilde{w} - c^2 \Delta \tilde{w} = 0 \quad \text{in } I_\epsilon \times \mathbb{R}^n, \tag{14}$$

where $I_\epsilon = (-T + 2\epsilon, T - 2\epsilon)$. Moreover, (13) implies that $\tilde{w} = 0$ and $\partial_\nu \tilde{w} = 0$ on $\Gamma(h - 2\epsilon)$. We will show below that $\tilde{w} \in C^\infty(I_\epsilon \times \mathbb{R}^n)$, and therefore we may apply Theorem 4.3 with $s = 1$ to obtain $\tilde{w} = 0$ and $\partial_t \tilde{w} = 0$ on $\{0\} \times \Omega(h - 2\epsilon)$. Letting $\psi \to \delta$ in the sense of distributions and $\epsilon \to 0$, we conclude that $w = 0$ and $\partial_t w = 0$ on $\{0\} \times \Omega(h)^{\text{int}}$. It remains to show that $\tilde{w}$ is smooth. Clearly $\tilde{w} \in C^\infty(I_\epsilon; H^s(\mathbb{R}^n))$ and (14) imply that $\Delta \tilde{w}(t) \in H^s(\mathbb{R}^n)$, $t \in \mathbb{R}$. Thus $\tilde{w}(t) \in H^{s+2}(\mathbb{R}^n)$, $t \in \mathbb{R}$, and we see that $\tilde{w}$ is smooth using an induction. $\square$

4.3. Recovery under the translation and separation conditions

To simplify the notation, we will assume below without loss of generality that $t_1 = 0$ and $x_1 = 0$.

Lemma 4.4. *Let $x_j \in \Omega$, $j = 1, 2, \ldots, J$ satisfy (SS) and define $\rho \in [0, 1)$ by (9). Let $r > 0$. Then for any $j, k = 1, \ldots, J$ and any $y \in B_r(x_j)$, there exists $x \in B_r(x_k)$, so that*

$$d(x, y) \leq \rho \left| t_j - t_k \right|.$$

Proof. Suppose first that $j < k$ and note that (SS) implies that

$$d(x_k, x_j) \leq d(x_k, x_{k-1}) + d(x_{k-1}, x_{k-2}) + \cdots + d(x_{j+1}, x_j) = \rho(t_k - t_j).$$

Combining this with an analogous computation in the case $j > k$ yields

$$d(x_k, x_j) \leq \rho|t_k - t_j|, \quad j, k = 1, \ldots, J. \tag{15}$$

Let x be the closest point in $B_r(x_k)$ to y. Then the geodesic from y to x hits $\partial B_r(x_k)$ normally by [32, Corollary 26], whence $d(y, x) = d(y, x_k) - d(x_k, x) = d(y, x_k) - R$. We conclude by observing that $d(y, x_k) \leq d(y, x_j) + d(x_j, x_k) \leq R + \rho |t_j - t_k|$. □

The recovery of the smooth part is based on finite speed of propagation and unique continuation as described in the following two lemmas respectively. Briefly, first we will show that there is a gap in time where only signals from the first source have arrived; this is illustrated in Figure 3. Then we use unique continuation to determine f_1 in part of S_1.

Lemma 4.5. *Let $x_j \in \Omega$, $j = 1, 2, \ldots, J$ satisfy (SS) and define $\rho \in [0, 1)$ and $R > 0$ by (9). Consider the solutions w_j, $j = 1, 2, \ldots, J$, of (4). If $(t, y) \in (0, T) \times \partial\Omega$ satisfies*

$$t \leq d(y, B_R(x_1)) + (1 - \rho)t_2,$$

then $\chi_{\{t \geq t_j\}} \partial_\nu^k w_j(t, y) = 0$ for all k and for all $j \geq 2$.

Proof. We write $B_j = B_R(x_j)$, $j = 1, \ldots, J$. Since $d(y, S_j) \geq d(y, B_j)$, by finite speed of propagation, it will be sufficient to show

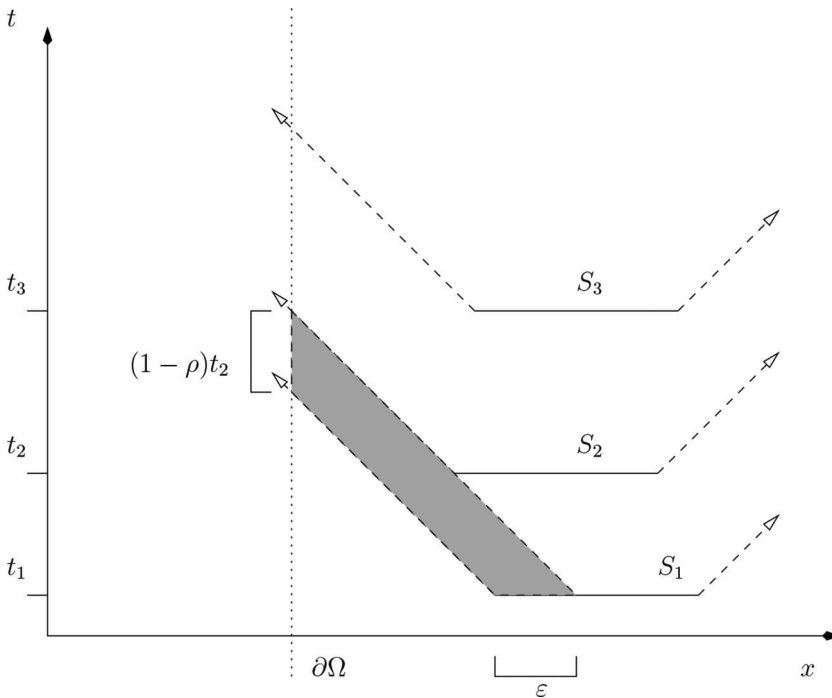$$d(y, B_j) \geq t - t_j, \quad t \geq t_j, j \geq 2. \tag{16}$$

Figure 3. The gray area is affected only by the first source. Here, $t_1 = 0$.

Let z be the closest point to y in B_j. By Lemma 4.4, there is $x \in B_1$ such that $d(z, x) \leq \rho t_j$. Thus

$$t - d(y, B_1) \leq (1 - \rho)t_2 \leq (1 - \rho)t_j \leq t_j - d(z, x).$$

Hence

$$t - t_j \leq d(y, B_1) - d(z, x) \leq d(y, x) - d(z, x) \leq d(y, z) = d(y, B_j).$$

$\square$

Lemma 4.6. *Let $x_j \in \Omega, j = 1, 2, \ldots, J$ satisfy (SS) and define $\rho \in [0, 1)$ and $R > 0$ by (9). We write*

$$\varepsilon_0 = (1 - \rho)t_2, \quad B_1 = B_R(x_1), \tag{17}$$

and let $\varepsilon \in (0, \varepsilon_0]$. If $T > \max_{y \in \partial\Omega} d(y, B_1) + \varepsilon$ then f_1 is uniquely determined by ΛF in the interior of the set

$$\Omega_\varepsilon = \left\{ x \in \Omega; \text{ there is } y \in \partial\Omega \text{ such that } d(x, y) \leq d(y, B_1) + \varepsilon \right\}.$$

Proof. By solving the exterior problem

$$\begin{cases} \partial_t^2 u - c(x)^2 \Delta u = 0 & \text{in } (0, T) \times \mathbb{R}^n \backslash \Omega, \\ u|_{x \in \partial\Omega} = \Lambda F & \text{in } (0, T) \times \partial\Omega, \\ u(0, \cdot) = \partial_t u(0, \cdot) = 0 & \text{in } \mathbb{R}^n \backslash \Omega, \end{cases}$$

we recover $\partial_\nu u|_{(0,T) \times \partial\Omega}$. We define $H_0 = (u, \partial_\nu u)$ and

$$H(t, y) = \begin{cases} H_0(t, y), & t \in (0, T) \\ -H_0(-t, y), & t \in (-T, 0), \end{cases} \quad y \in \partial\Omega.$$

We set $h(y) = d(y, B_1) + \varepsilon$, $y \in \partial\Omega$ and define $\Gamma(h)$ as in Theorem 4.3. Lemma 4.5 implies that $H = (w_1, \partial_\nu w_1)$ on $\Gamma(h) \cap (0, T) \times \partial\Omega$, and we have assumed that $\max_{y \in \partial\Omega} h(y) < T$. As $t_1 = 0$ and w_1 satisfies (4), w_1 is odd as a function of time. Therefore, $H = (w_1, \partial_\nu w_1)$ on $\Gamma(h)$, and Theorem 4.3 implies that $f_1 = \partial_t w_1(0, \cdot)$ is uniquely determined by H on the set $\Omega_\varepsilon^{\text{int}}$. $\square$

Proof of Theorem 2.5. We use the notations from Lemma 4.6. Recall that we have assumed $\varepsilon_0 > 2R$. We take $\varepsilon = \text{diam}(B_1) \leq 2R$ and observe that T satisfies the inequality in Lemma 4.6 by the assumption $T > t_J + \text{diam}(\Omega)$. Lemma 4.6 implies that f_1 is determined on Ω_ε and our choice of ε implies that $B_R(x_1) \subset \Omega_\varepsilon$. Thus f_1 is determined.

We solve the wave equation (1) with F replaced by $F_0 = \delta(t - t_1)f_1$. Then we can determine $\Lambda F_1 = \Lambda F - \Lambda F_0$ where $F_1 = \sum_{j=2}^{J} \delta(t - t_j)f_j$. We iterate the above steps to recover $f_2, \ldots, f_J$. $\square$

Remark 4.7. Let us consider again the partial data case in Remark 3.2. By that remark, we can recover the source times $t_j, j = 1, 2, \ldots, J$. Analogously to Lemma 4.6 and Theorem 2.5, it is possible to apply unique continuation to recover a part of f_1 and even the whole F if a strong enough separation condition is satisfied.

Lemma 4.8. *Suppose that $(\Omega, c^{-2} dx^2)$ is simple and define Ω_ε as in Lemma 4.6. Then*

$$\Omega_\varepsilon = (B_R(0) \backslash B_{R-\varepsilon}(0)) \cap \Omega.$$

Proof. Clearly $\Omega \cap B_{R-\varepsilon}(0)^{\mathrm{int}} \subset \Omega \backslash \Omega_\varepsilon$ even if simplicity is not assumed. Let $z \in \Omega \backslash \Omega_\varepsilon$ and choose $\xi \in S_0 \Omega$ and $s \geq 0$ such that $z = \exp_0(s\xi)$. It is sufficient to show that $z \in B_{R-\varepsilon}(0)^{\mathrm{int}}$. We define x and y by (12), i.e. $x = \exp_0(R\xi)$ and $y = \exp_0(\tau_0(\xi)\xi)$.

First, suppose that $s \geq R$. As z is in between x and y on the geodesic $t \mapsto \exp_0(t\xi)$, and as all the geodesics are distance minimizing on a simple manifold, we have

$$d(z, y) \leq d(x, z) + d(z, y) = d(x, y) = d(y, B_1),$$

which contradicts $z \in \Omega \backslash \Omega_\varepsilon$, and therefore, we have shown that $s < R$.

Next, as x is in between z and y on the geodesic $t \mapsto \exp_p(t\xi)$, we have

$$d(z, y) = d(z, x) + d(x, y) = d(z, x) + d(y, B_1).$$

Moreover, $z \in \Omega \backslash \Omega_\varepsilon$ implies that

$$d(y, B_1) < d(z, y) - \varepsilon.$$

Hence $\varepsilon < d(z, x) = R - s$, and therefore $s < R - \varepsilon$. Thus $z \in B_{R-\varepsilon}(0)^{\mathrm{int}}$. $\qquad\square$

Proof of Theorem 2.3. We choose $\epsilon = \min(\epsilon_0, R)$ and observe that T satisfies the inequality in Lemma 4.6 by (8). We recall the assumption that $S_j \subset \Omega$. By Lemmas 4.6 and 4.8, f_1 is uniquely determined on the set $B_R(0) \backslash B_{R-\varepsilon}(0)$. By (R1), the function f_j is obtained from f_1 through the translation $\exp_{x_j} \circ \mathcal{T}_{x_j} \circ \exp_0^{-1}$. This translation maps $B_R(0) \backslash B_{R-\varepsilon}(0)$ to

$$A_j := B_R(x_j) \backslash B_{R-\varepsilon}(x_j),$$

and therefore we can determine $f_j|_{A_j}$.

We solve the wave equation (1) with F replaced by $F_0(t, x) = \sum_{j=1}^J \delta(t - t_j) f_j|_{A_j}(x)$. Then we can determine $\Lambda F - \Lambda F_0 = \Lambda F_1$, where $F_1(t, x) = \sum_{j=1}^J \delta(t - t_j) \tilde{f}_j(x)$ and $\tilde{f}_j$ is the restriction of f_j on $B_{R-\varepsilon}(x_j)$. If $\epsilon = R$, then we have recovered F, otherwise we repeat the above construction starting from ΛF_1. This iteration allows us to decrease the radius R by $(1 - \rho)t_2$ in each step (Figure 4), and therefore it will terminate in a finite number of steps. $\qquad\square$

Proof of Theorem 2.4. As before, f_1 is uniquely determined on the set $B_R(0) \backslash B_{R-\varepsilon}(0)$. We may assume without the loss of generality that $\epsilon < R$. Let us denote by $B_r^E(x)$ the Euclidean ball of radius r centered at x. As the geodesic ball $B_{R-\varepsilon}(0)$ is contained in the Euclidean ball $B_{c^+(R-\varepsilon)}^E(0)$, we know f_1 outside $B_{c^+(R-\varepsilon)}^E(0)$. The translation assumption (E1) implies that f_j is known outside $B_{c^+(R-\varepsilon)}^E(x_j)$. This last ball is contained in the geodesic ball $B_{R^{(1)}}(x_j)$, where $R^{(1)} = \frac{c^+}{c^-}(R-\varepsilon)$. As above we may remove the contribution of the known part of the functions f_j from the data ΛF and iterate the construction.

We terminate the iteration if $R^{(n)} \leq \varepsilon$. Otherwise we set $R^{(n+1)} = \frac{c^+}{c^-}(R^{(n)} - \varepsilon)$ and reduce to the case $S_j \subset B_{R^{(n+1)}}(x_j)$. We have

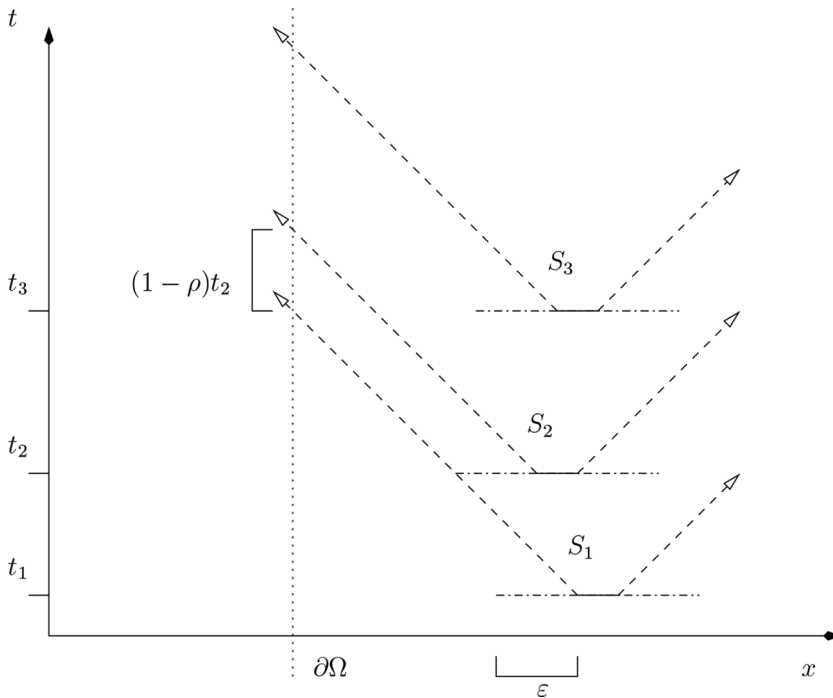$$R^{(n)} - R^{(n+1)} = \frac{c^+}{c^-}\left(\epsilon - \left(1 - \frac{c^-}{c^+}\right) R^{(n)}\right).$$

Figure 4. At each step of the iteration, the radius where f_1 is unknown and decreases by $(1 - \rho)t_2$.

The assumption (E2) implies that

$$\left(1 - \frac{c^-}{c^+}\right) R < (1 - \rho)t_2 = \epsilon.$$

Thus the sequence $R^{(n)}$ is decreasing and

$$R^{(n)} - R^{(n+1)} \geq \frac{c^+}{c^-}\left(\epsilon - \left(1 - \frac{c^-}{c^+}\right) R\right).$$

Thus each step of the iteration decreases the radius by an amount that is bounded from below by a strictly positive constant, and therefore the iteration terminates in a finite number of steps. □

5. Determining the translations from the supports S_j

Let us begin by considering the Euclidean translation condition (E1). Suppose that we know the sets $S_j, j = 1, 2, \ldots, J$. We define the center of mass

$$\tilde{x}_j = \frac{1}{|\Sigma_j|} \int_{\Sigma_j} x dx.$$

where $|\Sigma_j|$ is the Euclidean $n - 1$ dimensional volume of Σ_j, dx is the Euclidean surface measure on Σ_j and Σ_j is defined by (5). By (E1) the function f_j is obtained from f_1 through the translation $\mathcal{T}_j^E(x) = x + x_j - x_1$. Also the centers of mass are mapped through this translation, whence $\tilde{x}_j - \tilde{x}_1 = x_j - x_1$. Thus we can determine the translations $\mathcal{T}_j^E$ given the

supports S_j for all $j = 1, 2, \ldots, J$. When applying Theorem 2.4 to recover the source F, we may assume that $x_j = \tilde{x}_j$ since this amounts to replacing f with the translation $\tilde{f}(x) = f(x + \tilde{x})$, where $\tilde{x}$ is the center of mass of supp (f).

We turn now to the Riemannian translation condition (R1) and consider only the case $\dim(S_j) = n$. By (R1), the function f_j is obtained from f_1 through the translation $\mathcal{T}_j^R(x) = \exp_{x_j} \circ \mathcal{T}_{x_j} \circ \mathcal{T}_{x_1}^{-1} \circ \exp_{x_1}^{-1}$. We will give next a condition that guarantees that the translations $\mathcal{T}_j^R$ can be determined using centers of mass analgously to the Euclidean case.

Let κ and K be a lower bound for the injectivity radius and an upper bound for the sectional curvature of the Riemannian manifold $(\Omega, c^{-2} dx^2)$, respectively, and define

$$r_\Omega = \min\left\{ \kappa, \frac{\pi}{2\sqrt{K}} \right\}.$$

Suppose that $S \subset \mathbb{R}^n$ is measurable set that is contained in a geodesic ball $B(p, r) \subset \Omega$ where $p \in \Omega$ and $r < r_\Omega$. Then the function $\varrho_S(x) = \max_{y \in S} d(x, y)$ has a unique minimizer x_S (see [3] Theorem 2.1).

Let us write $S = \text{supp}(f)$ and denote by $|\xi|_g$ the norm of $\xi \in T_0\Omega$ with respect to the Riemannian metric $g = c^{-2} dx^2$. We suppose that there is $R \in (0, r_\Omega)$ such that

(i) $|\xi|_g \le R$ for all $\xi \in S$ and

(ii) there is $\xi_0 \in S_0\Omega$ such that $R\xi_0 \in S$ and $-R\xi_0 \in S$. $\qquad$ (R2)

The condition (R2) implies that there are two points on the boundary of S that are symmetric with respect to the origin.

Lemma 5.1. *Suppose that* (R1) *and* (R2) *hold. Then the minimizer* x_{S_j} *is* x_j.

Proof. For any $x \in \Omega$, the parallel translation $\mathcal{T}_x$ is a linear isometry, and if $\xi \in T_x\Omega$ satisfies $|\xi|_g \le \kappa$ and $\exp_x(\xi) \in \Omega$, then $d(\exp_x(\xi), x) = |\xi|_g$. Let $j = 1, \ldots, J$ and define $x^\pm = \exp_{x_j}(\pm R\mathcal{T}_{x_j}\xi_0)$. Then for all $x \in \Omega$

$$d(x^+, x) + d(x^-, x) \ge d(x^+, x^-) = 2R,$$

and $\varrho_{S_j}(x) \ge R$. On the other hand, $S_j \subset B(x_j, R)$. Hence x_j is a minimizer of ϱ_{S_j}. $\qquad \square$

6. Examples

The condition (ML1) can be seen as consisting of two requirements: first, that no outward propagating wavefront intersects any later wavefront, and second, that no inward propagating wavefront intersects any later wavefront. We show below that the first part of (ML1) is implied by (SS) under some further conditions.

Example 1. If $\Sigma_j = \partial B_r(x_j)$ (e.g., $S_j = B_r(x_j)$ or $S_j = \partial B_r(x_j)$), $j = 1, 2, \ldots, J$, for some $r > 0$, then (SS) implies the first part of (ML1).

Proof. To see this, note that the outgoing wavefront due to Σ_j at time t is $\partial B_{r+t-t_j}(x_j)$. Choose any $k > j$ and $x \in \overline{B_r(x_k)}$, by (SS), there is some $y \in B_r(x_j)$, so that $d(x, y) < \rho |t_j - t_k|$, and

further, $d(y, x_j) < r$ so that $d(x, x_j) < r + \rho(t_k - t_j)$ showing

$$x \in B_{r+\rho(t_k-t_j)}(x_j) \subset B_{r+t_k-t_j}(x_j)$$

so that the wavefront has already completely passed S_k at $t = t_k$. ☐

Example 2. Suppose that the Riemannian manifold $(\Omega, c^{-2}dx^2)$ is simple. If S_j are arbitrary, and (TS) is satisfied, then the first part of (ML1) is satisfied.

Proof. To demonstrate this claim, suppose that an outgoing ray from $x \in \Sigma_j$ intersects S_k at some point y at time t (if we can show intersections do not happen on the base manifold, then they do not happen in the cotangent bundle either). If $t < t_k$ then there is nothing to verify, so assume $t \geq t_k$. Then on one hand, $d(y, x) = t - t_j \geq t_k - t_j$, and on the other

$$d(y, x) \leq d(y, x_k) + d(x_k, x_j) + d(x_j, x) \leq d(x_k, x_j) + 2R \leq \rho(t_k - t_j) + 2R.$$

Then, by (TS),

$$2R < (1 - \rho)(t_{j+1} - t_j) \leq (1 - \rho)(t_k - t_j)$$

so that, finally

$$d(y, x) \leq \rho(t_k - t_j) + 2R < t_k - t_j$$

which is a contradiction. ☐

Further, (TS) implies (SS), so that if the second part of (ML1), (ML2), and (TS) are assumed, then all the hypotheses for Theorems 2.2 and 2.5 are satisfied, yielding a complete reconstruction.

For the next example, let r_c be the maximum r such that $B_r(x)$ is convex for every $x \in \Omega$. This is known as the convexity radius of Ω, and it is positive for any compact manifold (see [7], Proposition 95).

Example 3. If $\dim(S_j) = n$ and S_j are convex, and $t_J - t_1 < r_c$, then (SS) implies the first part of (ML1).

To see that convexity is essential in Example 3, consider Figure 5. Here, for a nonconvex "horseshoe" shaped S_1, a ray leaving the "bend" of the shoe intersects the "prong" at a time later than t_2. The proof of Example 3 is based on the following lemma.

Lemma 6.1. *Let C be a convex set in a Riemannian manifold (M, g), let $A = \partial C$, let $y \in M \backslash C$, and let σ be a geodesic from y to some point in $x \in A$ such that σ is normal to A at x and such that $d(x, y) < r_c$. Then σ minimizes the distance from y to C.*

Proof. For contradiction, assume there is some point $z \in A$ so that $d(y, z) < d(y, x)$.

Consider the totally geodesic hyperplane $\mathcal{S}$ tangent to A at x. Because C is strictly convex, it lies entirely on one side of $\mathcal{S}$; call this side H_1, and the other H_2 and note that both are convex. Let $B = B_{d(y,x)}(y)$; as a radial geodesic, σ is normal to ∂B at x, and thus $\mathcal{S}$ is tangent to B as well. Because $d(y, x) < r_c$, B is convex and must also lie entirely on one side of $\mathcal{S}$.
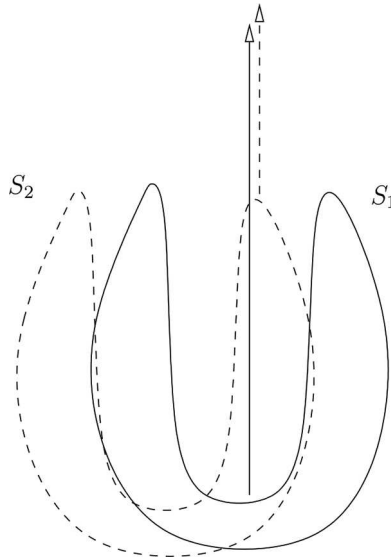
Figure 5. A nonconvex set.

For some $s_1 > d(y, x)$, $\sigma(s_1) \in H_1$, and for some $s_2 < d(y, x)$, $\sigma(s_2) \in H_2$. Thus we must have $y \in H_2$, otherwise σ is a geodesic that exits and then reenters H_1, violating convexity. Thus B and C lie on opposite sides of $\mathcal{S}$.

Therefore, since $B_{d(y,z)}(y) \subset B$, z cannot be in A, and we have a contradiction. $\qquad \square$

Proof of Example 3. Now, suppose that an outgoing ray from $x \in \partial S_j$ intersects S_k at some point y at time t (as before, it is sufficient to show intersections do not happen in the base manifold). If $t < t_k$, then there is nothing to verify, so assume $t \geq t_k$. Then $d(y, x) = t - t_j < r_c$, so $d(y, x) = d(y, S_j)$ by Lemma 6.1. On the other hand, $t - t_j \geq t_k - t_j > \rho(t_k - t_j)$ which violates (SS). $\qquad \square$

7. Genericity of the microlocal conditions

In this section, we show that both the assumptions (ML1) and (ML2) are generic.

To simplify the notation, we write $\Sigma = \Sigma_1$. Let $\mathcal{B} \subset C^\kappa(\overline{\Sigma})$ be a small neighbourhood of the origin, so that the function

$$x(y, h) = (h(y), y), \quad y \in \Sigma, \ h \in \mathcal{B},$$

takes values in the domain of the boundary normal coordinates of Σ. We will fix the smoothness index $\kappa \in \mathbb{N}$ below. Consider perturbations of Σ parametrized by $h \in \mathcal{B}$,

$$\Sigma(h) = \{x(y, h) \in \mathbb{R} \times \Sigma; \ y \in \Sigma\}.$$

Note that the conormal vectors of $\Sigma(h)$ at y are spanned by $\tilde{\nu}(y, h) = (1, -dh(y))$. We define the unit conormal vector field $\nu(y, h) = \tilde{\nu}(y, h)/|\tilde{\nu}(y, h)|_g$ where $|\cdot|_g$ denotes the norm with respect to the metric $g = c^{-2}dx^2$. Furthermore, we define

$$F : \Sigma \times \mathcal{B} \to S^*\mathbb{R}^n, \quad F(y, h) = (x(y, h), \nu(y, h)).$$

Here $S^*\mathbb{R}^n$ is the unit cosphere bundle with respect to the metric g. We choose the smoothness index κ, so that F is C^n-smooth.

Lemma 7.1. *For any $y \in \Sigma$, the differential of F is surjective from $T_{(y,0)}(\Sigma \times \mathcal{B})$ to $T_{(y,\nu(y,0))}S^*\mathbb{R}^n$.*

Proof. We use local coordinates on Σ near y. Consider a path $\gamma : [0,1] \to S^*\mathbb{R}^n$ such that $\gamma(0) = (y, \nu(y,0))$ and write in the boundary normal coordinates

$$\gamma(s) = (r(s), y(s), a(s), \eta(s)).$$

We define

$$h_s(z) = \chi(z)(r(s) - a^{-1}(s)\eta(s)(z - y(s))),$$

where χ is a smooth cutoff function satisfying $\chi = 1$ near y. Note that $r(0) = 0$, $\eta(0) = 0$, and $a(0) = 1$. Thus $h_s \in \mathcal{B}$ for small s. Moreover, $\tilde{\nu}(y(s), h_s) = a^{-1}(s)(a(s), \eta(s))$ and

$$|\tilde{\nu}(y(s), h_s)|_g = a^{-1}(s).$$

Hence $F(y(s), h_s) = \gamma(s)$ and $\dot{\gamma}(0)$ is in the range of dF at $(y,0)$.   □

We recall that a set is said to be meagre if it can be expressed as the union of countably many nowhere dense sets.

Lemma 7.2. *Consider the solutions w_j, $j = 1, \ldots, J$, of the equations (4). Then there is a meagre set $N \subset \mathcal{B}$ such that*

$$\mathrm{WF}(w_1) \cap \mathrm{WF}(w_j) = \emptyset, \quad j = 2, 3, \ldots, J.$$

when Σ_1 is replaced by any $\Sigma(h)$ with $h \in \mathcal{B} \backslash N$.

Proof. Let $j = 2, \ldots, J$, and define the projection

$$Z = \{(x, \xi) \in S^*\mathbb{R}^n; \ (t_k, x, 1, \xi) \in \mathrm{WF}(w_j)\},$$

Note that Z is $n - 1$ dimensional, since it can be written as

$$\{(\gamma(s), \dot{\gamma}(s)); \ \gamma = \gamma(\cdot; x, \pm\nu(x)), \ s = t_k - t_j, \ x \in \Sigma_j\},$$

where $\pm\nu(x)$ are the two unit conormal vectors of Σ_j at x. We use the notation $F_h(y) = F(y, h)$ for fixed $h \in \mathcal{B}$ and observe that $F_h(\Sigma)$ coincides with one of the two components of $N(\Sigma(h)) \cap S^*\mathbb{R}^n$. We will consider only this component, since the proof is analogous for the other component and the union of two meagre sets is also meagre. For the same reason it is enough to consider one j at a time.

As $\mathrm{WF}(w_j)$ and $\mathrm{WF}(w_k)$ are conical and invariant under the bicharacteristic flow, it is enough to show that $F_h(\Sigma) \cap Z = \emptyset$ for h in the complement of a meagre set, or in other words, in a residual set. The previous lemma implies that F is transversal to Z, and the parametric transversality theorem, see e.g., [1, Th. 3.6.19], implies that for h in a residual set, the map F_h is transversal to Z. By transversality, if there are $z \in Z$ and $y \in \Sigma$ such that $F_h(y) = z$, then $dF_h(T_y\Sigma) + T_zZ = T_zS^*\mathbb{R}^n$. But this is impossible since $\dim(T_y\Sigma) = n - 1 = \dim(T_zZ)$ and $2(n - 1) < 2n - 1 = \dim(T_zS^*\mathbb{R}^n)$.   □

By applying Lemma 7.2 with w_1 replaced by another w_k, $k = 2, \ldots, J$, we see that (ML1) holds generically. Let us now turn to (ML2). To simplify the notation suppose that (ML2) fails due to

$$\mathrm{WF}^+\,(w_1(t_0, \cdot)) = \widetilde{\mathrm{WF}^-}\,(w_2(t_0, \cdot))$$

for some $t_0 \in \mathbb{R}$. Then $N_1^+ \cap S^* \mathbb{R}^n = G_{t_0}(Z)$ where $Z = N_2^- \cap S^* \mathbb{R}^n$ and

$$G_{t_0} = \Gamma_{t_1 - t_0} \circ \widetilde{\Gamma}_{t_0 - t_2}.$$

Here Γ_s is the geodesic flow, that is,

$$\Gamma_s(x, \xi) = (\gamma(s; x, \xi), \dot{\gamma}(s; x, \xi)), \quad (x, \xi) \in S^* \mathbb{R}^n,$$

and $\sim$ is the reflection in the dual variable as in (7). We have $\Gamma_s(x, -\xi) = \Gamma_{-s}(x, \xi)$ and therefore

$$G_t = \Gamma_{-(t_1 - t)} \circ \Gamma_{t - t_2} = \Gamma_{2t - t_2 - t_1} = \Gamma_{2(t - t_0)} \circ G_{t_0}.$$

In the boundary normal coordinates of $\Sigma = \Sigma_1$, it holds for all t near t_0 that

$$G_t(Z) = \{F(y, h_t); \ y \in \Sigma\}$$

where $h_t(y) = 2(t - t_0)$, $y \in \Sigma$. Thus for all nonconstant functions $h \in \mathcal{B}$,

$$\mathrm{WF}^+\,(w_1(t, \cdot)) \neq \widetilde{\mathrm{WF}^-}\,(w_2(t, \cdot))$$

when Σ is replaced by $\Sigma(h)$ and t is near t_0. Clearly the constant functions are nowhere dense in $\mathcal{B}$. By repeating the above argument for other possible spurious parings, we see that (ML2) holds generically.

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