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Uniqueness for the electrostatic inverse boundary value problem with piecewise constant anisotropic conductivities

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Abstract

We discuss the inverse problem of determining the, possibly anisotropic, conductivity of a body $\Omega \subset \mathbb{R}^n$ when the so-called Neumann-to-Dirichlet map is locally given on a non-empty curved portion Σ of the boundary $\partial\Omega$. We prove that anisotropic conductivities that are *a priori* known to be piecewise constant matrices on a given partition of Ω with curved interfaces can be uniquely determined in the interior from the knowledge of the local Neumann-to-Dirichlet map.

Keywords: Calderòn's problem, electrical impedance tomography, direct current (DC) method, anisotropy

1. Introduction

The inverse problem of recovering the conductivity of a body by taking measurements of voltage and current on its surface is studied in the present paper. More specifically, the case when the conductivity is anisotropic and it is *a priori* known to be a piecewise-constant matrix on a given partition of a domain (the body under investigation) is considered. It is well known that in absence of internal sources, the electrostatic potential u in a conducting body, described by a domain $\Omega \subset \mathbb{R}^n$, is governed by the elliptic equation

$$\mathrm{div}(\sigma\nabla u) = 0 \qquad \text{in} \quad \Omega, \tag{1.1}$$

where the symmetric, positive definite matrix $\sigma = \sigma(x)$, $x \in \Omega$ represents the (possibly anisotropic) electric conductivity. The inverse conductivity problem consists of finding σ when the so called Dirichlet-to-Neumann (D-N) map

$$\Lambda_\sigma : u|_{\partial\Omega} \in H^{\frac{1}{2}}(\partial\Omega) \longrightarrow \sigma\nabla u \cdot \nu|_{\partial\Omega} \in H^{-\frac{1}{2}}(\partial\Omega)$$

is given for any $u \in H^1(\Omega)$ solution to (1.1). Here, ν denotes the unit outer normal to $\partial\Omega$. If measurements can be taken only on one portion Σ of $\partial\Omega$, then the relevant map is called the local D-N map (Λ_σ^Σ).

Different materials display different electrical properties, so that a map of the conductivity $\sigma(x)$, $x \in \Omega$ can be used to investigate internal properties of Ω. This problem has many important applications in fields such as geophysics, medicine and non-destructive testing of materials. The first mathematical formulation of the inverse conductivity problem is due to Calderón [C], where he addressed the problem of whether it is possible to determine the (isotropic) conductivity $\sigma = \gamma I$ by the D-N map. [C] opened the way to the solution to the uniqueness issue where one is asking whether σ can be determined by the knowledge of Λ_σ (or Λ_σ^Σ in the case of local measurements). We introduce the following function spaces

$$_0H^{\frac{1}{2}}(\partial\Omega) = \left\{ f \in H^{\frac{1}{2}}(\partial\Omega) | \int_{\partial\Omega} f = 0 \right\},$$

$$_0H^{-\frac{1}{2}}(\partial\Omega) = \left\{ \psi \in H^{-\frac{1}{2}}(\partial\Omega) | \langle \psi, 1 \rangle = 0 \right\}.$$

We observe that the D-N map Λ_σ maps onto $_0H^{-\frac{1}{2}}(\partial\Omega)$, and, when restricted to $_0H^{\frac{1}{2}}(\partial\Omega)$, is injective with bounded inverse called the Neumann-to-Dirichlet (N-D) map. The precise definitions of the D-N, N-D and their local versions will be given in section 2. For now, we simply recall that the N-D map associates to specified current densities supported on a portion $\Sigma \subset \partial\Omega$ the corresponding boundary voltages measured on the same portion Σ of $\partial\Omega$. For the applications of the inverse conductivity problem to the direct-current (DC) resistivity method that we have in mind, the choice of taking the surface measurements by means of the (local) N-D map over the (local) D-N map seems to be appropriate.

The case when measurements can be taken over the full boundary has been studied extensively in the past. Fundamental papers like [Ko-V1, Ko-V2, N, Sy-U1] and [Al] show that the isotropic case can be considered solved. More recently these uniqueness results have been extended in dimension $n \geqslant 3$ for conductivities in C^1 [Ha-T], for Lipschitz conductivities [Ca-R] and for conductivities in $W^{s,p}(\Omega) \nsubseteq W^{1,\infty}(\Omega)$ [Ha], again, while assuming full boundary data. The original uniqueness result by Sylvester and Uhlmann [Sy-U1] required the conductivity to be C^∞. For the two-dimensional case we refer to [Bro-U] and the breakthrough paper [As-P] where uniqueness has been proven for conductivities that are merely L^∞. We also recall the uniqueness results of Druskin who, independently from Calderón, dealt directly with the geophysical setting of the problem in [D1–D3]. His uniqueness result obtained in [D2] was for conductivities described by piecewise constant functions (see also [Al-V]). In the present paper, we consider conductivities that are piecewise constant matrices. We refer to [Bo, C-I-N] and [U] for an overview regarding the issues of uniqueness and reconstruction of the conductivity.

The problem of recovering the conductivity σ by local measurements has been treated more recently. Lassas and Uhlmann [La-U] recovered a connected compact real-analytic Riemannian manifold (M, g) with boundary by making use of the Green's function of the Laplace–Beltrami operator Δ_g; see also [La-U-T]. For the procedure of reconstructing the conductivity at the boundary by local measurements we refer to [Bro, NaT1, NaT2, K-Y]. An overview on reconstruction formulas for the conductivity and its normal derivative can be found in [NaT3]. For related results of uniqueness in the interior in the case of local boundary data, we refer to Bukhgeim and Uhlmann [B-U], Kenig, Sjöstrand and Uhlmann [Ke-S-U] and Isakov [Is1], and, for stability, Heck and Wang [He-W]. Results of stability for cases of piecewise constant conductivities and local boundary maps have also been obtained in [Al-V, Be-Fr] and [D].

The inverse problem with anisotropic conductivities, however, has remained open. Since Tartar's observation [Ko-V1] that any diffeomorphism of Ω which keeps the boundary points fixed has the property of leaving the D-N map unchanged, whereas σ is modified, different lines of research have been pursued. One direction has been to find the conductivity up to a diffeomorphism which keeps the boundary fixed (see [Le-U, Sy, N, La-U, La-U-T, Be] and [As-La-P]). Another direction has been the one to formulate suitable *a priori* assumptions (possibly fitting some physical context) which constrain the structure of the unknown anisotropic conductivity. For instance, one can formulate the hypothesis that the directions of anisotropy are known while some scalar space dependent parameter is not. Along this line of reasoning, we mention the results in [Ko-V1, Al, Al-G, Al-G1, G-Li, G-S] and [Li]. The case when $n = 2$ and the anisotropic conductivity is assumed to be divergence free has been treated in [Al-C].

Here, we follow this second direction by *a priori* assuming that the conductivity is piecewise constant in a known finite partition of the domain (a segmentation), whereas the constant, matrix-valued, conductivities in each subdomain are unknown. An additional (apparently necessary) assumption that we pose is that contiguous subdomains of the partition can be joined by *curved* smooth surfaces and also that the boundary portion Σ where measurements are collected also contains a *curved* portion of a surface. Under such assumptions we show, theorem 2.1, that a local boundary map uniquely determines the conductivity, also in the interior. For the sake of concreteness we focus our analysis on the local N-D map. But it will be evident from the proof that also other choices of the boundary maps could be treated.

We give an outline of the underlying ideas in our approach. As is well known, [B-G-M, U], the solutions to equation (1.1) are the harmonic functions on the Riemannian manifold $\{\Omega, g\}$ where the metric g is linked to the conductivity σ through the relation

$$g = (\det \sigma)^{\frac{1}{n-2}} \sigma^{-1}.$$

We obtain, in lemma 3.5, that, under a few regularity assumptions, one can uniquely determine from the knowledge of the local N-D map near a point $P \in \partial\Omega$, the tangential part of $g(P)$, that is, the $(n-1) \times (n-1)$ minor of $g(P)$ relative to the tangent (hyper)plane to $\partial\Omega$ at P. Incidentally, we recall that a similar result was already contained in proposition 1.3 of Lee and Uhlmann [Le-U]; under stronger regularity assumptions, it was shown that the same minor is determined by the D-N map. If the local N-D map is known on a non-flat portion Σ of $\partial\Omega$ and σ is constant nearby, then we have enough different tangent planes to completely recover g, and hence σ, see lemma 3.6. Thus, the crucial point in our argument stands in proving uniqueness at the boundary of the anisotropic conductivity. It may be surprising that such a result appears after more than three decades since the seminal result of boundary determination by Kohn and Vogelius in the isotropic case [Ko-V, Ko-V1]. However, the present approach is rather different from those originally used with highly oscillating boundary data [Ko-V], the analysis of the symbol of the D-N map as a pseudodifferential operator [Sy-U] and the use of singular solutions [Is, Al]. The approach used here, relies on an accurate inspection of the asymptotics of the Neumann kernel (which enters in the integral representation of the N-D map). Such an asymptotics has indeed its roots in the potential theoretic approach to studying elliptic equations [Mi], but its application to inverse boundary value problems is, to the best of our knowledge, a novelty.

The proof is then completed by an iteration argument and by the use of the unique continuation property. For this step (from the boundary to the interior) a well-known approach that could be used, and is well rooted in fundamental papers [Ko-V2, Is], is the one based on the application of the Runge approximation property [Lax]. However, we chose here to develop arguments of a slightly different character, which have a more constructive flavor and have the potential of being translated into stability estimates.

Finally, in example 4.2, which is a variation of the celebrated Tartar's example, [Ko-V1], we show that the N-D map for the half space is *not* sufficient to uniquely determine a *constant* anisotropic conductivity. Thus, this example provides a strong indication that indeed, flat boundary and interfaces may constitute an obstruction to uniqueness and thus our assumptions on curved interfaces and boundary are well motivated. For other kinds of examples of nonuniqueness we may refer to [G-La-U1] and [G-La-U2].

We gave an overview of the introduction and application of the DC method (or ERT) in geophysical exploration and geothermal prospecting in a companion paper [Al-dH-G-S 1]. As early as 1920, Conrad Schlumberger [Sc] recognized that anisotropy may affect geological formations' DC electrical properties. Anisotropic effects when measuring electromagnetic fields in geophysical applications have been studied ever since. From an inverse problems perspective, it is interesting that Maillet and Doll [M-Do] already identified obstructions to recovering an anisotropic resistivity from (boundary) data. Individual minerals are typically anisotropic but rocks composed of them can appear to be isotropic. Simpson and Tommasi [Si-T] discussed the application of effective medium models to calculate the (degree of) anisotropy in electrical conductivity in an aggregate with non-random crystallographic orientations. In fact, there are many heterogeneous material configurations in Earth's sedimentary basins that possibly lead to anisotropy [Ne-S]. It might be that there are some preferred directions in the subsurface rocks, or some preferred orientation of grains in the sediments. Fine layering or a pronounced strike direction can lead to an effective anisotropy. For example, alternations of sandstone and shales can cause hydrocarbon reservoir anisotropy, but anisotropy in shale-free sandstones can occur as well [Ken-H]. Resistivity anisotropy has also been measured in volcanic reservoir rock [No].

In view of practical constraints on the data acquisition, DC resistivity methods are limited to probing Earth's (upper) crust. Resolving conductive structures to depths of the upper mantle requires magnetotelluric (MT) data. The analysis of the MT inverse boundary value problem associated with the low-frequency Maxwell equations will be presented in a separate paper. Most minerals in Earth's deeper interior (lower crust, upper mantle and transition zone) have been shown to have anisotropic conductivities that are sensitive not only to temperature, but also to hydrogen (water) content, major element chemistry and oxygen fugacity [Ka-W]. Consequently, there is a potential to infer the distribution of these chemical factors (as well as temperature) from the study of electrical conductivities. Here, the influence of partial melting[4] needs to be accounted for. Indeed, to infer the water distribution in Earth's mantle, electrical conductivity plays a primary role [Ka][5].

The paper is organized as follows. Our main assumptions and our main result (theorem 2.1) are contained in section 2, whereas section 3 contains some preliminary results. The proof of theorem 2.1, that is, the proof of the unique determination of the piecewise constant anisotropic conductivity from the knowledge of the local N-D map, is contained in section 4. We emphasize that the consideration of the local N-D map, rather than the local D-N map, is motivated by the application of this inverse problem to the DC resistivity method in geophysical prospecting.

[4] Melts in general have higher electrical conductivity than minerals. This is essentially due to the high diffusion coefficients of charged species in melts [Ho]. As a consequence, the presence of partial melt will contribute to relatively high electrical conductivity.

[5] Hydrogen (water) has an important influence on rheological properties [Ka-J] and melting relationship ([Ku-Syo-Ak, I]) that control the dynamics and evolution of our planet.

2. Main result

2.1. Notation and definition

In several places in this manuscript it will be useful to single out one coordinate direction. To this purpose, the following notations for points $x \in \mathbb{R}^n$ will be adopted. For $n \geqslant 3$, a point $x \in \mathbb{R}^n$ will be denoted by $x = (x', x_n)$, where $x' \in \mathbb{R}^{n-1}$ and $x_n \in \mathbb{R}$. Moreover, given a point $x \in \mathbb{R}^n$, we shall denote with $B_r(x), B'_r(x)$ the open balls in $\mathbb{R}^n, \mathbb{R}^{n-1}$ respectively centred at x with radius r and by $Q_r(x)$ the cylinder $B'_r(x') \times (x_n - r, x_n + r)$. We shall denote $\mathbb{R}^n_+ = \{(x', x_n) \in \mathbb{R}^n | x_n > 0\}$, $B^+_r = B_r \cap \mathbb{R}^n_+$, where we understand $B_r = B_r(0)$ and $Q_r = Q_r(0)$.

We shall assume throughout that Ω is a bounded domain with Lipschitz boundary, see e.g. [A-F, 4.9].

Definition 2.1. Let Ω be a domain in $\mathbb{R}^n$. Given α, $\alpha \in (0, 1)$, we say that a portion Σ of $\partial\Omega$ is of class $C^{1,\alpha}$ if for any $P \in \Sigma$ there exists a rigid transformation of $\mathbb{R}^n$ under which we have $P = 0$ and

$$\Omega \cap Q_{r_0} = \{x \in Q_{r_0} \,|\, x_n > \varphi(x')\},$$

where φ is a $C^{1,\alpha}$ function on B'_{r_0} satisfying

$$\varphi(0) = |\nabla_{x'}\varphi(0)| = 0.$$

Definition 2.2. Given Σ as above, we shall say that such a portion of a surface is non-flat (and equivalently the function φ) if, there exists $P \in \Sigma$ such that, considering the reference system and the function φ as above, we have that φ is not identically zero near $P = 0$.

2.1.1. The Dirichlet-to-Neumann map. We begin with defining the D-N map. We denote by Sym_n the class of $n \times n$ symmetric real valued matrices. Let Ω be a domain in $\mathbb{R}^n$ with Lipschitz boundary $\partial\Omega$ and assume that $\sigma \in L^\infty(\Omega\,, Sym_n)$ satisfies the ellipticity condition

$$\lambda^{-1}|\xi|^2 \leqslant \sigma(x)\xi \cdot \xi \leqslant \lambda|\xi|^2, \quad \begin{array}{l} \textit{for almost every } x \in \Omega, \\ \textit{for every } \xi \in \mathbb{R}^n. \end{array} \tag{2.1}$$

We shall also denote by $\langle \cdot, \cdot \rangle$ the $L^2(\partial\Omega)$-pairing between $H^{\frac{1}{2}}(\partial\Omega)$ and its dual $H^{-\frac{1}{2}}(\partial\Omega)$.

Definition 2.3. The Dirichlet-to-Neumann (D-N) map associated with σ is the operator

$$\Lambda_\sigma : H^{\frac{1}{2}}(\partial\Omega) \longrightarrow H^{-\frac{1}{2}}(\partial\Omega) \tag{2.2}$$

defined by

$$\langle \Lambda_\sigma f,\ g \rangle \ = \ \int_\Omega \sigma(x)\nabla u(x) \cdot \nabla\varphi(x)\, \mathrm{d}x, \tag{2.3}$$

for any $f, g \in H^{\frac{1}{2}}(\partial\Omega)$, where $u \in H^1(\Omega)$ is the weak solution to

$$\begin{cases} \operatorname{div}(\sigma(x)\nabla u(x)) = 0, & \text{in} \quad \Omega, \\ u = f, & \text{on} \quad \partial\Omega, \end{cases}$$

and $\varphi \in H^1(\Omega)$ is any function such that $\varphi|_{\partial\Omega} = g$ in the trace sense.

Note that, by (2.3), it is easily verified that Λ_σ is selfadjoint. Given $\sigma^{(i)} \in L^\infty(\Omega, Sym_n)$, satisfying (2.1), for $i = 1, 2$, we recall Alessandrini's identity (see [Al, b, p 253])

$$\langle (\Lambda_{\sigma^{(1)}} - \Lambda_{\sigma^{(2)}}) f_1, f_2 \rangle = \int_\Omega \left(\sigma^{(1)}(x) - \sigma^{(2)}(x) \right) \nabla u_1(x) \cdot \nabla u_2(x), \tag{2.4}$$

for any $f_i \in H^{\frac{1}{2}}(\partial\Omega)$, $i = 1, 2$ and $u_i \in H^1(\Omega)$ being the unique weak solution to the Dirichlet problem

$$\begin{cases} \mathrm{div}(\sigma^{(i)}(x)\nabla u_i(x)) = 0, & \text{in} \quad \Omega, \\ u_i = f_i, & \text{on} \quad \partial\Omega. \end{cases}$$

We define now the local N-D map.

2.1.2. The Neumann-to-Dirichlet map. We consider the following function spaces

$$_0H^{\frac{1}{2}}(\partial\Omega) = \left\{ f \in H^{\frac{1}{2}}(\partial\Omega) | \int_{\partial\Omega} f \ = 0 \right\},$$

$$_0H^{-\frac{1}{2}}(\partial\Omega) = \left\{ \psi \in H^{-\frac{1}{2}}(\partial\Omega) | \ \langle \psi, \ 1 \rangle = 0 \right\}.$$

As previously observed, the D-N map Λ_σ maps onto $_0H^{-\frac{1}{2}}(\partial\Omega)$, and, when restricted to $_0H^{\frac{1}{2}}(\partial\Omega)$, it is injective with bounded inverse. Then we can define the global Neumann-to-Dirichlet map as follows.

Definition 2.4. The Neumann-to-Dirichlet (N-D) map associated with σ,

$$\mathcal{N}_\sigma : \ _0H^{-\frac{1}{2}}(\partial\Omega) \longrightarrow \ _0H^{\frac{1}{2}}(\partial\Omega)$$

is given by

$$\mathcal{N}_\sigma = \left(\Lambda_\sigma|_{_0H^{\frac{1}{2}}(\partial\Omega)} \right)^{-1}. \tag{2.5}$$

Note that $\mathcal{N}_\sigma$ can also be characterized as the selfadjoint operator satisfying

$$\langle \psi, \ \mathcal{N}_\sigma \psi \rangle \ = \ \int_\Omega \sigma(x) \nabla u(x) \cdot \nabla u(x) \, \mathrm{d}x, \tag{2.6}$$

for every $\psi \in \ _0H^{-\frac{1}{2}}(\partial\Omega)$, where $u \in H^1(\Omega)$ is the weak solution to the Neumann problem

$$\begin{cases} \mathrm{div}(\sigma \nabla u) = 0, & \text{in} \quad \Omega, \\ \sigma \nabla u \cdot \nu|_{\partial\Omega} = \psi, & \text{on} \quad \partial\Omega, \\ \int_{\partial\Omega} u \ = 0. \end{cases} \tag{2.7}$$

Given $\sigma^{(i)} \in L^\infty(\Omega, Sym_n)$, satisfying (2.1), for $i = 1, 2$, the following identity can be recovered from (2.4)

$$\langle \sigma^{(1)} \nabla u_1 \cdot \nu, (\mathcal{N}_{\sigma^{(2)}} - \mathcal{N}_{\sigma^{(1)}}) \, \sigma^{(2)} \nabla u_2 \cdot \nu \rangle = \int_\Omega \left(\sigma^{(1)}(x) - \sigma^{(2)}(x) \right) \nabla u_1(x) \cdot \nabla u_2(x), \tag{2.8}$$

for any $u_i \in H^1(\Omega)$ weak solution to

$$\mathrm{div}(\sigma^{(i)}(x) \nabla u_i(x)) = 0, \qquad \text{in} \quad \Omega, \tag{2.9}$$

for $i = 1, 2$.

Now we introduce the local version of the N-D map. Let Σ be an open portion of $\partial\Omega$ and let $\Delta = \partial\Omega \setminus \overline{\Sigma}$. We introduce the subspace of $H^{\frac{1}{2}}(\partial\Omega)$,

$$H^{\frac{1}{2}}_{co}(\Delta) = \left\{ f \in H^{\frac{1}{2}}(\partial\Omega) \mid \mathrm{supp}(f) \subset \Delta \right\}.$$

We denote by $H^{\frac{1}{2}}_{00}(\Delta)$ the closure in $H^{\frac{1}{2}}(\partial\Omega)$ of the space $H^{\frac{1}{2}}_{co}(\Delta)$ and we introduce

$$_0H^{-\frac{1}{2}}(\Sigma) = \left\{ \psi \in \, _0H^{-\frac{1}{2}}(\partial\Omega) | \, \langle \psi, f \rangle = 0, \quad \text{for any } f \in H^{\frac{1}{2}}_{00}(\Delta) \right\}, \qquad (2.10)$$

that is the space of distributions $\psi \in H^{-\frac{1}{2}}(\partial\Omega)$ which are supported in $\overline{\Sigma}$ and have zero average on $\partial\Omega$. The local N-D map is then defined as follows.

Definition 2.5. The local Neumann-to-Dirichlet map associated with σ, Σ is the operator $\mathcal{N}^{\Sigma}_{\sigma} : \, _0H^{-\frac{1}{2}}(\Sigma) \longrightarrow \left(_0H^{-\frac{1}{2}}(\Sigma) \right)^* \subset \, _0H^{\frac{1}{2}}(\partial\Omega)$ given by

$$\langle \mathcal{N}^{\Sigma}_{\sigma} \, \varphi, \, \psi \rangle = \langle \mathcal{N}_{\sigma} \, \varphi, \, \psi \rangle, \qquad (2.11)$$

for every $\varphi, \psi \in \, _0H^{-\frac{1}{2}}(\Sigma)$.

Given $\sigma^{(i)} \in L^{\infty}(\Omega \, , Sym_n)$, satisfying (2.1), for $i = 1, 2$, we also recover from (2.4)

$$\left\langle \psi_1, \left(\mathcal{N}^{\Sigma}_{\sigma^{(2)}} - \mathcal{N}^{\Sigma}_{\sigma^{(1)}} \right) \psi_2 \right\rangle = \int_{\Omega} \left(\sigma^{(1)}(x) - \sigma^{(2)}(x) \right) \nabla u_1(x) \cdot \nabla u_2(x), \qquad (2.12)$$

for any $\psi_i \in \, _0H^{-\frac{1}{2}}(\Sigma)$, for $i = 1, 2$ and $u_i \in H^1(\Omega)$ being the unique weak solution to the Neumann problem

$$\begin{cases} \mathrm{div}(\sigma^{(i)} \nabla u_i) = 0, & \text{in} \quad \Omega, \\ \sigma^{(i)} \nabla u_i \cdot \nu|_{\partial\Omega} = \psi_i, & \text{on} \quad \partial\Omega, \\ \int_{\partial\Omega} u_i \, = 0. & \end{cases} \qquad (2.13)$$

2.2. *The a priori assumptions*

2.2.1. *Assumptions pertaining to the domain partition.*

1. $\Omega \subset \mathbb{R}^n$ is a bounded domain, with $n \geqslant 3$.
2. $\partial\Omega$ is of Lipschitz class.
3. We fix an open non-empty subset Σ of $\partial\Omega$ (where the measurements in terms of the local N-D map are taken).
4. There exists $N \in (\mathbb{N} \setminus \{0\})$ such that

   $$\overline{\Omega} = \bigcup_{j=1}^{N} \overline{D_j},$$

   where D_j, $j = 1, \ldots, N$ are known open sets of $\mathbb{R}^n$, satisfying the conditions below.

   (a) D_j, $j = 1, \ldots, N$ are connected and pairwise nonoverlapping.
   (b) ∂D_j, $j = 1, \ldots, N$ are of Lipschitz class.

(c) There exist α, $\alpha \in (0,1)$ and one region, say D_1, such that $\partial D_1 \cap \Sigma$ contains a non flat $C^{1,\alpha}$ portion Σ_1.

(d) For every $i \in \{2,\ldots,N\}$ there exists $j_1,\ldots,j_K \in \{1,\ldots,N\}$ such that

$$D_{j_1} = D_1, \qquad D_{j_K} = D_i \tag{2.14}$$

and such that for every $i = 1,\ldots,K$

$$\left(\bigcup_{k=1}^{i} \overline{D_{j_k}}\right)^{\circ} \quad \text{and} \quad \Omega \setminus \left(\bigcup_{k=1}^{i} \overline{D}_{j_k}\right)$$

are Lipschitz domains.

In addition we assume that there exists α, $\alpha \in (0,1)$, such that for every $k = 2,\ldots,K$, $\partial D_{j_k} \cap \partial D_{j_{k-1}}$ contains a non flat $C^{1,\alpha}$ portion Σ_k (for the time being we agree that $D_{j_0} = \mathbb{R}^n \setminus \Omega$), such that

$$\Sigma_k \subset \Omega.$$

More specifically we assume that for every $k = 2,\ldots,K$ there exists $P_k \in \Sigma_k$ and a rigid transformation of coordinates under which we have $P_k = 0$ and

$$\begin{aligned} \Sigma_k \cap Q_{r_0/3} &= \{x \in Q_{r_0/3} | x_n = \varphi_k(x')\} \\ D_{j_k} \cap Q_{r_0/3} &= \{x \in Q_{r_0/3} | x_n > \varphi_k(x')\} \\ D_{j_{k-1}} \cap Q_{r_0/3} &= \{x \in Q_{r_0/3} | x_n < \varphi_k(x')\}, \end{aligned} \tag{2.15}$$

where φ_k is a non flat $C^{1,\alpha}$ function on $B'_{r_o/3}$ satisfying

$$\varphi_k(0) = |\nabla \varphi_k(0)| = 0.$$

2.2.2. Assumption pertaining to the conductivity. We assume that the conductivity σ is of type

$$\sigma(x) = \sum_{j=1}^{N} \sigma_j \chi_{D_j}(x), \qquad x \in \Omega, \tag{2.16}$$

where $\sigma_j \in Sym_n$ are positive definite constant matrices, satisfying the uniform ellipticity condition

$$\lambda^{-1}|\xi|^2 \leqslant \sigma_j \xi \cdot \xi \leqslant \lambda |\xi|^2, \qquad \text{for every } \xi \in \mathbb{R}^n, \tag{2.17}$$

for $j = 1,\ldots,N$, and D_j, $j = 1,\ldots,N$ are the subdomains introduced in section 2.2.1.

2.3. Global uniqueness

Our main result is stated below.

Theorem 2.1. *Let Ω, D_j, $j = 1,\ldots,N$ and Σ be a domain, N subdomains of Ω and a portion of $\partial\Omega$ as in section 2.2.1 respectively and let $\sigma^{(i)}$, $i = 1, 2$ be two conductivities of type*

$$\sigma^{(i)}(x) = \sum_{j=1}^{N} \sigma_j^{(i)} \chi_{D_j}(x) \qquad x \in \Omega,\ i = 1, 2, \tag{2.18}$$

where $\sigma_j^{(i)} \in Sym_n$ *are positive definite constant matrices, satisfying the uniform ellipticity condition (2.17), for* $j = 1, \ldots, N$. *If*

$$\mathcal{N}^{\Sigma}_{\sigma^{(1)}} = \mathcal{N}^{\Sigma}_{\sigma^{(2)}},$$

then

$$\sigma^{(1)} = \sigma^{(2)}, \qquad \text{in} \quad \Omega. \tag{2.19}$$

3. The Neumann kernel

From now on we shall denote by $\sigma(x) = \{\sigma_{ij}(x)\}_{i,j=1,\ldots,n}$, $x \in \Omega$ a symmetric, positive definite matrix-valued function satisfying (2.17) and denote by L the operator

$$L = \mathrm{div}\,(\sigma\nabla\cdot)\,. \tag{3.1}$$

We shall also introduce the matrix

$$g = (\det \sigma)^{\frac{1}{n-2}}\,\sigma^{-1}. \tag{3.2}$$

Remark 3.1. If we endow the open set Ω with the Riemannian metric g, then

$$\frac{1}{\sqrt{\det g}} L = \Delta_g,$$

that is, up to the factor $\frac{1}{\sqrt{\det g}}$, the operator L can be viewed as the Laplace–Beltrami operator for the Riemannian manifold $\{M, g\}$, see for instance [B-G-M, U]. We emphasize that, being $n > 2$, the knowledge of σ is equivalent to the knowledge of g.

We digress and consider the operator (3.1) on the half space $\mathbb{R}^n_+$ with σ constant. We denote by

$$\Pi_n = \{x = (x', x_n) \in \mathbb{R}^n \mid x_n = 0\}$$

the hyperplane in $\mathbb{R}^n$ of points with vanishing n^{th} coordinate. From now on we will denote by $\xi \cdot \rho$ the Euclidean scalar product of vectors $\xi, \rho \in \mathbb{R}^n$.

Note that when σ is constant, the same is true for g. We shall denote by $g_{(n-1)}$ the $(n-1) \times (n-1)$ submatrix of g obtained by removing the n^{th} row and column from g.

Lemma 3.2. *Let* N_σ *be the Neumann kernel for the operator (3.1), with constant* $\sigma \in Sym_n$, *on the half space* $\mathbb{R}^n_+$ *and pole at the boundary. That is, for any* $y' \in \Pi_n$, *let* N_σ *be the distributional solution to*

$$\begin{cases} L\,N_\sigma(\cdot, y') = 0, & \text{in} \quad \mathbb{R}^n_+, \\ \sigma\nabla N_\sigma(\cdot, y)\cdot\nu = \delta(\cdot - y'), & \text{on} \quad \Pi_n, \\ N_\sigma(x, y') \to 0, & \text{as} \quad |x| \to \infty. \end{cases}$$

For every $x \in \mathbb{R}^n_+$ *and* $y' \in \Pi_n$ *we have*

$$N_\sigma(x, y') = 2C_n\,(g(x-y')\cdot(x-y'))^{\frac{2-n}{2}}\,, \tag{3.3}$$

where

$$C_n = \frac{1}{n(n-2)\omega_n}, \tag{3.4}$$

with ω_n denoting the volume of the unit ball in $\mathbb{R}^n$. In particular, if $N_\sigma(x', y')$ is known for every $x', y' \in \Pi_n$ then $g_{(n-1)}$ is uniquely determined.

Proof. The verification of (3.3) is straightforward when σ is the identity. The general case follows straightforwardly through a linear change of variables. Furthermore, when $x', y' \in \Pi_n$, we may write

$$N_\sigma(x', y') = 2C_n \, (g(x'-y')\cdot(x'-y'))^{\frac{2-n}{2}} \tag{3.5}$$

or as is the same

$$g(x'-y')\cdot(x'-y') = \left(\frac{N_\sigma(x', y')}{2C_n}\right)^{\frac{2}{2-n}}, \quad \text{for all} \quad x', y' \in \Pi_n.$$

Consequently $g_{(n-1)}$ is uniquely determined by $N_\sigma(x', y')$, $x', y' \in \Pi_n$. □

We shall also introduce the Neumann kernel N_σ^Ω for the boundary value problem associated with the operator (3.1) and Ω by defining it, for any $y \in \Omega$, $N_\sigma^\Omega(\cdot, y)$ to be the distributional solution to

$$\begin{cases} L\, N_\sigma^\Omega(\cdot, y) = -\delta(\cdot - y), & \text{in} \quad \Omega \\ \sigma \nabla N_\sigma^\Omega(\cdot, y)\cdot\nu = -\frac{1}{|\partial\Omega|}, & \text{on} \quad \partial\Omega. \end{cases}$$

Note that N_σ^Ω is uniquely determined up to an additive constant. For simplicity we impose the normalization

$$\int_{\partial\Omega} N_\sigma^\Omega(\cdot, y)\, \mathrm{d}S(\cdot) = 0.$$

With this convention we obtain by Green's identities that

$$N_\sigma^\Omega(x, y) = N_\sigma^\Omega(y, x), \qquad \text{for all} \quad x, y \in \Omega, \quad x \neq y. \tag{3.6}$$

Remark 3.3. $N_\sigma^\Omega(x, y)$ extends continuously up to the boundary $\partial\Omega$ (provided that $x \neq y$) and in particular, when $y \in \partial\Omega$, it solves

$$\begin{cases} L\, N_\sigma^\Omega(\cdot, y) = 0, & \text{in} \quad \Omega \\ \sigma \nabla N_\sigma^\Omega(\cdot, y)\cdot\nu = \delta(\cdot - y) - \frac{1}{|\partial\Omega|}, & \text{on} \quad \partial\Omega. \end{cases}$$

Theorem 3.4. *Let $y \in \partial\Omega$ and assume that there exists a neighborhood $\mathcal{U}$ of y such that $\partial\Omega \cap \mathcal{U}$ is a portion of class $C^{1,\alpha}$, with $0 < \alpha < 1$, of $\partial\Omega$ and σ in (3.1) is such that $\sigma \in C^\alpha(\mathcal{U} \cap \overline{\Omega})$. Then the Neumann kernel $N_\sigma^\Omega(\cdot, y)$ satisfies*

$$N_\sigma^\Omega(x, y) = 2C_n \left(\det(\sigma(y))\right)^{-1/2} \left(\sigma^{-1}(y)(x-y)\cdot(x-y)\right)^{\frac{2-n}{2}} + O(|x-y|^{2-n+\alpha}), \tag{3.7}$$

as $x \to y$, $x \in \overline{\Omega} \setminus \{y\}$ and C_n is the constant given in (3.4).

Proof. This result is typical and possibly well known. We refer to [Mi, section 1] and [Mit-T, 1.31–1.33] for the case $\sigma \in C^{\alpha}(\Omega)$, with $\partial\Omega$ of class $C^{1,\alpha}$. We sketch a proof for the sake of completeness. We represent $\Sigma = \partial\Omega \cap \mathcal{U}$ according to definition 2.1, and assume without loss of generality that $y = 0$. Let $r > 0$ be such that $\overline{B_r} \subset \mathcal{U}$. For any $\psi \in C_0^{0,1}(B_r)$, we have

$$\int_{\Omega\cap B_r} \sigma(x)\nabla_x N_\sigma^\Omega(x,0)\cdot\nabla_x\psi(x)\,\mathrm{d}x = \psi(0) - \frac{1}{|\partial\Omega|}\int_{\partial\Omega\cap B_r}\psi(x)\,\mathrm{d}S(x). \tag{3.8}$$

We introduce the change of coordinates $z = z(x)$ $(x = x(z))$

$$\begin{cases} z' = x' \\ z_n = x_n - \varphi(x') \end{cases}.$$

We have

$$z = x + O(|x'|^{1+\alpha}) \tag{3.9}$$

and also, setting $J = \frac{\partial z}{\partial x}$,

$$J = I + O(|x'|^{\alpha}). \tag{3.10}$$

Next, we define

$$\widetilde{\sigma}(z) = \left(\frac{1}{\det(J)} J\sigma J^T\right)(x(z)) \tag{3.11}$$

$$\widetilde{N}(z) = N_\sigma^\Omega(x(z),0)\,. \tag{3.12}$$

We obtain

$$\int_{\{z_n>0\}} \widetilde{\sigma}(z)\nabla_z\widetilde{N}(z)\cdot\nabla_z\psi(x(z))\,\mathrm{d}z = \psi(0) - \frac{1}{|\partial\Omega|}\int_{\Pi_n}\psi(z',0)\sqrt{1+|\nabla_{z'}\varphi|^2}\,\mathrm{d}z'. \tag{3.13}$$

We denote

$$q(z') = \frac{1}{|\partial\Omega|}\sqrt{1+|\nabla_{z'}\varphi|^2}$$

which is bounded. We have

$$\widetilde{\sigma}(z) = \sigma(0) + O(|z'|^{\alpha}). \tag{3.14}$$

We let N_0 denote the Neumann function for $\mathbb{R}^n_+$ with $\sigma = \sigma(0)$ and write

$$R(z) = \widetilde{N}(z) - N_0(z,0). \tag{3.15}$$

We have

$$\begin{aligned} &\int_{\{z_n>0\}} \widetilde{\sigma}(0)\nabla_z R(z)\cdot\nabla_z\psi(x(z))\,\mathrm{d}z \\ &= \int_{\{z_n>0\}} (\widetilde{\sigma}(0)-\widetilde{\sigma}(z))\,\nabla_z\widetilde{N}(z)\cdot\nabla_z\psi(x(z))\,\mathrm{d}z - \int_{\Pi_n}\psi(z',0)q(z')\,\mathrm{d}z'. \end{aligned} \tag{3.16}$$

Hence, for a sufficiently small $\rho > 0$, we have

$$\begin{cases} \mathrm{div}_z\,(\sigma(0)\nabla_z R(z)) = \mathrm{div}_z\left((\sigma(0)-\widetilde{\sigma}(z))\nabla_z \widetilde{N}(z)\right) & \text{in} \quad B_\rho^+, \\ \sigma(0)\nabla_z R(z)\cdot\nu = (\sigma(0)-\widetilde{\sigma}(z))\nabla_z\widetilde{N}(z)\cdot\nu - q(z') & \text{in} \quad B_\rho\cap\Pi_n. \end{cases}$$

We recall that

$$|N_\sigma^\Omega(x,0)| \leqslant C|x|^{2-n}, \qquad \text{for every} \quad x\in\Omega, \tag{3.17}$$

where $C > 0$ is a constant that only depends on ellipticity and on the Lipschitz regularity of $\partial\Omega$ (see e.g. [Ke-P]). Next using the local regularity of σ and of $\Sigma \subset \partial\Omega$ we also obtain

$$|\nabla_x N_\sigma^\Omega(x,0)| \leqslant C|x|^{1-n}, \qquad \text{for every} \quad x\in B_\rho\cap\Omega. \tag{3.18}$$

Consequently

$$|R(z)| + |z|\,|\nabla_z R(z)| \leqslant C \qquad \text{for every} \quad z\in\partial B_\rho\cap\mathbb{R}^n_+. \tag{3.19}$$

By Green's identities, for every $w \in B_\rho^+$ we obtain

$$\begin{aligned} R(w) = &-\int_{B_\rho^+} R(z)\,\mathrm{div}_z\,(\sigma(0)\nabla_z N_0(z,w))\;\mathrm{d}z \\ = &-\int_{\partial B_\rho^+}\left(R(z)\sigma(0)\nabla_z N_0(z,w)\cdot\nu - N_0(z,w)\sigma(0)\nabla_z R(z)\cdot\nu\right)\mathrm{d}S(z) \\ &-\int_{B_\rho^+} N_0(z,w)\,\mathrm{div}_z\,(\sigma(0)\nabla_z R(z))\;\mathrm{d}z \\ = &-\int_{\partial B_\rho^+}\left(R(z)\sigma(0)\nabla_z N_0(z,w)\cdot\nu - N_0(z,w)\sigma(0)\nabla_z R(z)\cdot\nu\right)\mathrm{d}S(z) \\ &-\int_{\partial B_\rho^+} N_0(z,w)\,(\sigma(0)-\widetilde{\sigma}(z))\,\nabla_z\widetilde{N}(z,w)\cdot\nu\;\;\mathrm{d}S(z) \\ &+\int_{B_\rho^+}(\sigma(0)-\widetilde{\sigma}(z))\,\nabla_z N_0(z,w)\cdot\nabla_z\widetilde{N}(z)\;\;\mathrm{d}z. \end{aligned} \tag{3.20}$$

If we split $\partial B_\rho^+ = \left(\partial B_\rho\cap\mathbb{R}^n_+\right)\cup(B_\rho\cap\Pi_n)$, we obtain

$$\begin{aligned} R(w) = &-\int_{\partial B_\rho\cap\mathbb{R}^n_+}\left(R(z)\sigma(0)\nabla_z N_0(z,w)\cdot\nu - N_0(z,w)\sigma(0)\nabla_z R(z)\cdot\nu\right)\mathrm{d}S(z) \\ &-\int_{\partial B_\rho\cap\mathbb{R}^n_+} N_0(z,w)\,(\sigma(0)-\widetilde{\sigma}(z))\,\nabla_z\widetilde{N}(z,w)\cdot\nu\;\;\mathrm{d}S(z) \\ &-\int_{B_\rho\cap\Pi_n} N_0(z',w)q(z')\,\mathrm{d}z' \\ &+\int_{B_\rho^+}(\sigma(0)-\widetilde{\sigma}(z))\,\nabla_z N_0(z,w)\cdot\nabla_z\widetilde{N}(z)\;\;\mathrm{d}z. \end{aligned} \tag{3.21}$$

Upon taking $|w| < \frac{\rho}{2}$, all the boundary integrals in (3.21) are uniformly bounded. In view of (3.14) and of (3.18), the volume integral in (3.21) can be estimated as follows

$$
\begin{aligned}
&\left|\int_{B_\rho^+} (\sigma(0) - \widetilde{\sigma}(z))\, \nabla_z N_0(z,w) \cdot \nabla_z \widetilde{N}(z)\ \mathrm{d}z\right| \\
&\leqslant C \int_{B_\rho^+} |z'|^\alpha\, |z-w|^{1-n}\, |z|^{1-n}\, \mathrm{d}z \leqslant C|w|^{2-n+\alpha},
\end{aligned}
\tag{3.22}
$$

hence $|R(z)| \leqslant C|z|^{2-n+\alpha}$ on B_ρ^+ and recalling that $|z| = O(|x|)$ the thesis follows. □

Therefore, we have

Lemma 3.5. *If $y' \in \partial\Omega$ and there is a neighborhood $\mathcal{U}$ of y' such that $\partial\Omega \cap \mathcal{U}$ is a portion of $\partial\Omega$ of class $C^{1,\alpha}$ and L is the operator (3.1), with coefficients matrix $\sigma \in C^\alpha(\mathcal{U} \cap \overline{\Omega})$, with $0 < \alpha < 1$, then the knowledge of $N_\sigma^\Omega(x, y')$, for every $x \in \partial\Omega \cap \mathcal{U}$ uniquely determines*

$$
g_{(n-1)}(y') = \{g(y')v_i \cdot v_j\}_{i,j=1,\ldots,(n-1)}\,, \tag{3.23}
$$

where $v_1, \ldots, v_{n-1}$ is a basis for $T_{y'}(\partial\Omega)$, the tangent plane to $\partial\Omega$ at y'.

Proof. Without loss of generality, we choose a coordinate system at $y' \in \partial\Omega$ such that $y' = 0$ and the tangent plane to $\partial\Omega$ at y' is $T_0(\partial\Omega) = \Pi_n$. For any $\xi \in \Pi_n$, $|\xi| = 1$, we choose $x' = r\xi$, with r small and denote $x = (x', \varphi(x')) \in \partial\Omega$, then by (3.7)

$$
\lim_{r\to 0} N_\sigma^\Omega(x, y')\, r^{\frac{n-2}{2}} = 2C_n\, (g(y')\xi \cdot \xi)^{\frac{2-n}{2}}\,,
$$

for all $\xi \in \Pi_n$, $|\xi| = 1$. Hence $g_{(n-1)}(y')$ is uniquely determined. □

Lemma 3.6. *Let Ω be a domain in $\mathbb{R}^n$ with boundary $\partial\Omega$ of Lipschitz class and let Σ be an open portion of $\partial\Omega$ of class $C^{1,\alpha}$ and non flat near some point $y_0' \in \Sigma$. If $\sigma \in L^\infty(\Omega\,, Sym_n)$ satisfies (2.1) and it is constant near y_0' and Σ, then the knowledge of $N_\sigma^\Omega(x', y')$, for every $x', y' \in \Sigma$ uniquely determines $\sigma(y_0')$.*

Proof. We denote by $\{e_1, \ldots, e_n\}$ the canonical basis in $\mathbb{R}^n$. We assume, without loss of generality, that $y_0' = 0 \in \Sigma$, that the tangent space to $\partial\Omega$ at $0 \in \Sigma$ is $T_0(\partial\Sigma) = \Pi_n =< e_1, \ldots, e_{n-1} >$ and the outer unit normal to $\partial\Omega$ at 0 is $-e_n$. For any $P \in \partial\Omega$, we will denote by $\nu(P)$ the outer unit normal to $\partial\Omega$ at P ($\nu(0) = -e_n$). If Σ is not flat near 0, then there are points $P \in \Sigma$ nearby such that $\nu(P)$ slightly deflects from $\nu(0) = -e_n$, therefore without loss of generality, we can assume that there exists a point $P \in \Sigma$ and some $\varepsilon \neq 0$ such that

$$
\nu(P) = \frac{1}{\sqrt{1+\varepsilon^2}}\,(-e_n + \varepsilon e_{n-1})\,. \tag{3.24}
$$

Depending on the geometry of Σ near 0, there is an alternative:

(a) The deflection of ν is everywhere in the e_{n-1} direction.
(b) There are points $\widetilde{P} \in \Sigma$ near 0 in which the deflection of ν is in a direction independent of e_{n-1} and without loss of generality, we can assume that there is a point $\widetilde{P} \in \Sigma$ and some $\alpha, \beta \in \mathbb{R}$, with $\alpha \neq 0$ such that

$$
\nu(\widetilde{P}) = \frac{1}{\sqrt{1+\alpha^2+\beta^2}}\,(-e_n + \alpha e_{n-2} + \beta e_{n-1})\,. \tag{3.25}
$$

Next, we show that in either cases (a) and (b), $g(0)$ (hence $\sigma(0)$) can be uniquely determined. We denote by

$$g = g(0)$$

and start with case (a). In this case an orthonormal basis for the tangent space $T_P(\Sigma)$ is given by

$$\left\{e_1, \ldots e_{n-2}, \frac{1}{\sqrt{1+\varepsilon^2}}\, e_{n-1} + \frac{\varepsilon}{\sqrt{1+\varepsilon^2}}\, e_n\right\}. \tag{3.26}$$

Suppose $\varepsilon > 0$. By continuity, we can find a continuous path $Q = Q(t)$, for $0 < t < \varepsilon$ along Σ such that $Q(0) = 0$, $Q(\varepsilon) = P$, $g(Q(t)) = g$, $0 < t < \varepsilon$ and such that an orthonormal basis for the tangent space $T_{Q(t)}(\Sigma)$ is given by

$$\left\{e_1, \ldots e_{n-2}, \frac{1}{\sqrt{1+t^2}}\, e_{n-1} + \frac{t}{\sqrt{1+t^2}}\, e_n\right\}. \tag{3.27}$$

Recalling that by lemma 3.5 we know

$$gv_i \cdot v_j, \qquad i,j = 1, \ldots, n-1, \tag{3.28}$$

for all v_i, $i = 1, \ldots, n-1$, forming a basis for $T_{Q(t)}\Sigma$, for any t, $0 < t < \varepsilon$, we have that the following functions

$$ge_i \cdot \left(\frac{1}{\sqrt{1+t^2}}\, e_{n-1} + \frac{t}{\sqrt{1+t^2}}\, e_n\right), \tag{3.29}$$

$$g\left(\frac{1}{\sqrt{1+t^2}}\, e_{n-1} + \frac{t}{\sqrt{1+t^2}}\, e_n\right) \cdot \left(\frac{1}{\sqrt{1+t^2}}\, e_{n-1} + \frac{t}{\sqrt{1+t^2}}\, e_n\right) \tag{3.30}$$

are known for any $i = 1, \ldots, n-1$ and any t, $0 < t < \varepsilon$. From (3.29) we obtain that the function

$$g_{i,\,n-1} + tg_{i,\,n} \tag{3.31}$$

is known for any any t, $0 < t < \varepsilon$, for any $i = 1, \ldots, n-2$ and hence $g_{i,\,n}$ is known for any $i = 1, \ldots, n-2$. From (3.30) we obtain that the polynomial

$$g_{n-1,\,n-1} + 2tg_{n-1,\,n} + t^2 g_{n,n} \tag{3.32}$$

is known for any t, $0 < t < \varepsilon$, hence all of its coefficients are known, in particular $g_{n-1,\,n}$ and $g_{n,n}$ are known too, therefore the full matrix g is determined in case (a).

Next, we consider case (b). For $\widetilde{P}$ near 0, we have that

$$g(\widetilde{P}) = g$$

and that $g_{i,j}$ is known for any $i,j = 1, \ldots, n-1$ by lemma 3.5. $g_{i,n}$ is also known for $i = 1, \ldots, n-2$ by recalling that the following scalar product

$$ge_i \cdot \left(\frac{1}{\sqrt{1+\varepsilon^2}}\, e_{n-1} + \frac{t}{\sqrt{1+\varepsilon^2}}\, e_n\right)$$

is known. To determine the remaining entries $g_{n-1,n}$, $g_{n,n}$ of the matrix g, we note that a basis for the tangent space $T_{\widetilde{P}}\Sigma$ is given by

$$\{e_1, \dots e_{n-3}, e_{n-2} + \alpha e_n, e_{n-1} + \beta e_n\}\,. \tag{3.33}$$

The following expressions

$$g\,(e_{n-2} + \alpha e_n) \cdot (e_{n-2} + \alpha e_n)\,, \tag{3.34}$$

$$g\,(e_{n-1} + \beta e_n) \cdot (e_{n-2} + \alpha e_n) \tag{3.35}$$

are known and from (3.34) and (3.35) we recover that the following expressions

$$g_{n-2,n-2} + 2\alpha g_{n-2,n} + \alpha^2 g_{n,n}, \tag{3.36}$$

$$g_{n-1,n-2} + \beta g_{n,n-2} + \alpha g_{n-1,n} + \alpha\beta g_{n,n} \tag{3.37}$$

are known too. From (3.36), recalling that $g_{n-2,n-2}, g_{n-2,n}$ are known and that $\alpha \neq 0$, we determine $g_{n,n}$. From (3.37), recalling that

$$g_{n-1,n-2}, g_{n,n-2}, g_{n,n}$$

are known and again that $\alpha \neq 0$, we determine $g_{n-1,n}$, hence the matrix g is completely determined in this case too. □

Definition 3.1. Given distinct points $x, y, w, z \in \Sigma$, we define

$$K_\sigma(x, y, w, z) = N_\sigma(x, y) - N_\sigma(x, w) - N_\sigma(z, y) + N_\sigma(z, w). \tag{3.38}$$

Note that, fixing $w, z \in \Sigma$, K_σ, as a function of x, y, has the same asymptotic behaviour of $N_\sigma(x, y)$ as $x \to y$.

Remark 3.7. It is well known that the knowledge of the full N-D map is equivalent to the knowledge of the boundary values of the Neumann kernel. It can also be verified that the local knowledge of the kernel implies knowing the local N-D map. Here we make precise the adjustments needed in the local determination of the kernel from the knowledge of the local map. The following lemma states that from $\mathcal{N}_\sigma^\Sigma$ one can determine locally $N_\sigma(x, y)$ up to a bounded function which is the sum of two terms $N_\sigma(x, w)$, $N_\sigma(z, y) - N_\sigma(z, w)$, one depending on x only and the other depending on y only.

Lemma 3.8. *$\mathcal{N}_\sigma^\Sigma$ is known if and only if K_σ is known for any $x, y, w, z \in \Sigma$.*

Proof. For any $\varphi, \psi \in C_0^{0,1}(\Sigma) \cap_0 H^{-\frac{1}{2}}(\Sigma)$ we have

$$\langle \psi, \mathcal{N}_\sigma^\Sigma \varphi \rangle = \int_\Sigma \psi(\xi) \mathrm{d}S(\xi) \int_\Sigma N_\sigma(\xi, \eta)\varphi(\eta)\mathrm{d}S(\eta) \tag{3.39}$$

$$= \int_{\Sigma\times\Sigma} N_\sigma(\xi, \eta)\psi(\xi)\varphi(\eta)\mathrm{d}S(\xi)\, \mathrm{d}S(\eta). \tag{3.40}$$

Note that the right hand side of

$$N_\sigma(\xi, \eta) - K_\sigma(\xi, \eta, w, z) = N_\sigma(\xi, w) + N_\sigma(z, \eta) - N_\sigma(z, w) \tag{3.41}$$

is a sum of terms which depend on at most one of the two variables ξ and η. Recalling that φ, ψ have zero average it follows that $N_\sigma(\xi,\eta) - K_\sigma(\xi,\eta,w,z)$ is orthogonal to $\psi(\xi)\varphi(\eta)$ in $L^2(\Sigma\times\Sigma)$, therefore (3.40) leads to

$$\langle \psi, \mathcal{N}_\sigma^\Sigma \varphi\rangle = \int_{\Sigma\times\Sigma} K_\sigma(\xi,\eta,w,z)\psi(\xi)\varphi(\eta)\,\mathrm{d}S(\xi)\,\mathrm{d}S(\eta). \tag{3.42}$$

Hence K_σ uniquely determines $\mathcal{N}_\sigma^\Sigma$. Vice versa, we pick

$$\psi(\xi) = \delta_\varepsilon(\xi;x) - \delta_\varepsilon(\xi;z),$$
$$\varphi(\eta) = \delta_\varepsilon(\eta;y) - \delta_\varepsilon(\eta;w),$$

where δ_ε are approximate Dirac's delta functions on Σ centered on the second argument. From (3.40), by letting $\varepsilon \to 0$ we can determine

$$K_\sigma(x,y,w,z),$$

which concludes the proof. □

4. Proof of the main result

Proof of theorem 2.1. Without loss of generality, we can assume that

$$\Sigma = \Sigma_1.$$

Let $\sigma^{(i)}$, $i=1,2$, be two conductivities of type (2.18) satisfying (2.17). If

$$\mathcal{N}_{\sigma^{(1)}}^{\Sigma_1} = \mathcal{N}_{\sigma^{(2)}}^{\Sigma_1},$$

then

$$\sigma^{(1)} = \sigma^{(2)}, \qquad \text{in} \quad D_1. \tag{4.1}$$

We shall proceed by induction. Let D_K be a subdomain of Ω, with $K \neq 1$ and recall that there exist $j_1,\dots,j_K \in \{1,\dots,N\}$ such that

$$D_{j_1} = D_1, \dots D_{j_K} = D_K,$$

with $D_{j_1},\dots D_{j_K}$ satisfying assumption 4(d). For simplicity, we rearrange the indices of these subdomains so that the above mentioned chain is simply denoted by $D_1,\dots,D_K, K \leqslant N$. We assume that

$$\sigma^{(1)} = \sigma^{(2)}, \qquad \text{in} \quad D_i, \qquad \text{for every } i, \quad 1 \leqslant i \leqslant K \tag{4.2}$$

and show that

$$\sigma^{(1)} = \sigma^{(2)}, \qquad \text{in} \quad D_{K+1} \text{ too}.$$

We shall set

$$D = \left(\bigcup_{i=1}^K \overline{D_i}\right)^\circ; \qquad E = \Omega \setminus \overline{D}.$$

We shall denote by $\mathcal{N}_{\sigma^{(i)}}^{\Sigma_{K+1}}$ the local N-D map for the domain E relative to the conductivity $\sigma^{(i)}$ and localized on Σ_{K+1}, for $i = 1, 2$.

Claim 4.1. If $\mathcal{N}_{\sigma^{(1)}}^{\Sigma_1} = \mathcal{N}_{\sigma^{(2)}}^{\Sigma_1}$ and $\sigma^{(1)} = \sigma^{(2)}$ in D then $\mathcal{N}_{\sigma^{(1)}}^{\Sigma_{K+1}} = \mathcal{N}_{\sigma^{(2)}}^{\Sigma_{K+1}}$.

Proof of claim 4.1. Here we shall adapt some arguments already used in [Al-K]. Recall that up to a rigid transformation of coordinates we can assume that

$$P_1 = 0 \qquad ; \qquad (\mathbb{R}^n \setminus \Omega) \cap B_{r_0} = \{(x', x_n) \in B_{r_0} \mid x_n < \varphi(x')\},$$

where φ is a Lipschitz function such that

$$\varphi(0) = 0 \qquad \text{and} \qquad ||\varphi||_{C^{0,1}(B'_{r_0})} \leqslant Lr_0.$$

Denoting by

$$D_0 = \left\{ x \in (\mathbb{R}^n \setminus \Omega) \cap B_{r_0} \,\middle|\, |x_i| < \frac{2}{3}r_0,\ i = 1, \ldots, n-1,\ \left| x_n - \frac{r_0}{6} \right| < \frac{5}{6}r_0 \right\},$$

it turns out that the augmented domain $\Omega_0 = \Omega \cup D_0$ is of Lipschitz class with constants $\frac{r_0}{3}$ and $\widetilde{L}$, where $\widetilde{L}$ depends on L only. For any number $r \in \left(0, \frac{2}{3}r_0\right)$ we also denote

$$(D_0)_r = \{x \in D_0 \mid \mathrm{dist}(x, \Omega) > r\}.$$

For $i = 1, 2$ we consider the operator $L_i = \mathrm{div}(\sigma^{(i)}\nabla\cdot)$ in Ω and extend $\sigma^{(i)}$ to $\widetilde{\sigma}^{(i)}$ on Ω_0, by setting $\widetilde{\sigma}^{(i)}|_{D_0} = I$, where I denotes the $n \times n$ identity matrix. For $y \in \Omega_0$ we define the modified Neumann kernel $\widetilde{N}_{\sigma^{(i)}}$ as the solution to

$$\begin{cases} L_i \widetilde{N}^{\Omega}_{\widetilde{\sigma}^{(i)}}(\cdot, y) = -\delta(x - y), & \text{in} \quad \Omega_0 \\ \widetilde{\sigma}^{(i)} \nabla \widetilde{N}^{\Omega}_{\widetilde{\sigma}^{(i)}} \cdot \nu = 0, & \text{on} \quad \partial\Omega_0 \cap \partial\Omega \\ \widetilde{\sigma}^{(i)} \nabla \widetilde{N}^{\Omega}_{\widetilde{\sigma}^{(i)}} \cdot \nu = -\frac{1}{|\partial\Omega_0 \setminus \bar{\Omega}|}, & \text{on} \quad \partial\Omega_0 \setminus \bar{\Omega}. \end{cases}$$

Here we convene to normalize $\widetilde{N}^{\Omega}_{\widetilde{\sigma}^{(i)}}$, by prescribing

$$\int_{\partial\Omega_0} \widetilde{N}^{\Omega}_{\widetilde{\sigma}^{(i)}}(\cdot, y)\, \mathrm{d}S(\cdot) = 0.$$

Again, with this choice we obtain

$$\widetilde{N}^{\Omega}_{\widetilde{\sigma}^{(i)}}(x, y) = \widetilde{N}^{\Omega}_{\widetilde{\sigma}^{(i)}}(y, x), \qquad \text{for all} \quad x, y \in \Omega_0, \quad x \neq y. \tag{4.3}$$

From now on we will simplify our notation by denoting

$$\widetilde{N}^{\Omega}_{\widetilde{\sigma}^{(i)}} = \widetilde{N}^{(i)}.$$

Given $\psi \in C^{0,1}(\partial E)$, with $\mathrm{supp}\psi \subset \Sigma^{K+1}$ and $\int_{\partial E} \eta = 0$, we let $u^{(i)}$ solve

$$\begin{cases} L_i u^{(i)} = 0, & \text{in} \quad E \\ \sigma^{(i)} \nabla u \cdot \nu = \psi, & \text{on} \quad \partial E. \end{cases}$$

We consider a bounded extension operator

$$T : H^{\frac{1}{2}}(\partial E \cap \Omega) \longrightarrow H^1(\Omega),$$

such that, given $f \in H^{\frac{1}{2}}(\partial E \cap \Omega)$, we have

$$Tf\big|_{\Sigma_1} = 0.$$

We denote

$$\bar{u}^{(i)} = \begin{cases} u^{(i)}, & \text{in} \quad E \\ T\left(u^{(i)}\big|_{\partial E\cap\Omega}\right), & \text{in} \quad D. \end{cases}$$

Clearly $\bar{u}^{(i)} \in H^1(\Omega)$. For $x \in E$ we have

$$\begin{aligned} u^{(i)}(x) &= -\int_\Omega \bar{u}^{(i)}(y)\,\mathrm{div}_y\left(\sigma^{(i)}(y)\nabla_y\widetilde{N}^{(i)}(y,x)\right)\,\mathrm{d}y \\ &= -\int_{\partial\Omega} \bar{u}^{(i)}(y)\,\sigma^{(i)}(y)\nabla_y\widetilde{N}^{(i)}(y,x)\cdot\nu\,\mathrm{d}S(y) \\ &\quad + \int_\Omega \sigma^{(i)}(y)\nabla_y\bar{u}^{(i)}(y)\cdot\nabla_y\widetilde{N}^{(i)}(y,x)\,\mathrm{d}y \\ &= \int_E \sigma^{(i)}(y)\nabla_y\bar{u}^{(i)}(y)\cdot\nabla_y\widetilde{N}^{(i)}(y,x)\,\mathrm{d}y \\ &\quad + \int_D \sigma^{(i)}(y)\nabla_y\bar{u}^{(i)}(y)\cdot\nabla_y\widetilde{N}^{(i)}(y,x)\,\mathrm{d}y \\ &= \int_{\Sigma_{K+1}} \psi\widetilde{N}^{(i)}(y,x)\,\mathrm{d}S(y) + \int_D \sigma^{(i)}(y)\nabla_y\bar{u}^{(i)}(y)\cdot\nabla_y\widetilde{N}^{(i)}(y,x)\,\mathrm{d}y. \end{aligned} \tag{4.4}$$

By differentiating under the integrals and by using Fubini, we form

$$\begin{aligned} &\nabla_x u^{(1)}(x)\cdot\nabla_x u^{(2)}(x) \\ &= \int_{\Sigma_{K+1}\times\Sigma_{K+1}} \psi(y)\psi(z)\nabla_x\widetilde{N}^{(1)}(y,x)\cdot\nabla_x\widetilde{N}^{(2)}(z,x)\,\mathrm{d}y\,\mathrm{d}z \\ &+ \int_{\Sigma_{K+1}\times D} \psi(y)\sigma^{(2)}_{lk}(z)\partial_{z_l}\bar{u}^{(2)}(z)\partial_{z_k}\left(\nabla_x\widetilde{N}^{(1)}(y,x)\cdot\nabla_x\widetilde{N}^{(2)}(z,x)\right)\,\mathrm{d}y\,\mathrm{d}z \\ &+ \int_{D\times\Sigma_{K+1}} \psi(z)\sigma^{(1)}_{lk}(y)\partial_{y_l}\bar{u}^{(1)}(z)\partial_{y_k}\left(\nabla_x\widetilde{N}^{(2)}(z,x)\cdot\nabla_x\widetilde{N}^{(1)}(y,x)\right)\,\mathrm{d}y\,\mathrm{d}z \\ &+ \int_{D\times D} \sigma^{(2)}_{lk}(z)\partial_{z_l}\bar{u}^{(2)}(z)\sigma^{(1)}_{nm}(y)\partial_{y_n}\bar{u}^{(1)}(z)\partial_{z_k}\partial_{y_m}\left(\nabla_x\widetilde{N}^{(2)}(z,x)\cdot\nabla_x\widetilde{N}^{(1)}(y,x)\right)\,\mathrm{d}y\,\mathrm{d}z. \end{aligned}$$

Here, and in what follows, the summation convention over repeated indices is used. We define for $y,z \in D\cup D_0$

$$S(y,z) = \int_E \left(\sigma^{(1)}(x) - \sigma^{(2)}(x)\right)\nabla_x\widetilde{N}^{(1)}(y,x)\cdot\nabla_x\widetilde{N}^{(2)}(z,x)\,\mathrm{d}x. \tag{4.5}$$

For any $y, z \in \left(\overline{D} \cup \overline{D}_0\right)^{\circ}$ we verify that

$$\begin{aligned} &\mathrm{div}_y \left(\sigma^{(1)}(y)\nabla_y S(y,z)\right) = 0,\\ &\mathrm{div}_z \left(\sigma^{(2)}(z)\nabla_z S(y,z)\right) = 0. \end{aligned} \tag{4.6}$$

Moreover,

$$S(y,z) = \int_{\Omega} \left(\sigma^{(1)}(x) - \sigma^{(2)}(x)\right) \nabla_x \widetilde{N}^{(1)}(y,x) \cdot \nabla_x \widetilde{N}^{(2)}(z,x), \tag{4.7}$$

because $\sigma^{(1)} = \sigma^{(2)}$ on D by assumption. For $y, z \in D_0$, being these singular points outside Ω, by the identity (2.12) we obtain

$$S(y,z) = \left\langle \sigma^{(1)}\nabla\widetilde{N}^{(1)}(y,\cdot)\cdot\nu, \left(\mathcal{N}^{\Sigma_1}_{\sigma^{(2)}} - \mathcal{N}^{\Sigma_1}_{\sigma^{(1)}}\right) \sigma^{(2)}\nabla\widetilde{N}^{(2)}(y,\cdot)\cdot\nu \right\rangle = 0.$$

We recall that by the $C^{1,\alpha}$ regularity of the interfaces Σ_{j_k} within D, $S(y,z)$ satisfies the unique continuation property in each variable $y, z \in \left(\overline{D} \cup \overline{D}_0\right)^{\circ}$. Hence,

$$S(y,z) = 0, \qquad \text{for any} \quad y, z \in D. \tag{4.8}$$

As a consequence, we obtain

$$\begin{aligned} &\int_E \left(\sigma^{(1)}(x) - \sigma^{(2)}(x)\right) \nabla_x u^{(1)}(x) \cdot \nabla_x u^{(2)}(x)\, \mathrm{d}x\\ &= \int_{\Sigma_{K+1}\times\Sigma_{K+1}} \psi(y)\psi(z)S(y,z)\, \mathrm{d}y\, \mathrm{d}z\\ &+ \int_{\Sigma_{K+1}\times D} \psi(y)\sigma^{(2)}_{lk}(z)\partial_{z_l}\overline{u}^{(2)}(z)\partial_{z_k}S(y,z)\, \mathrm{d}y\, \mathrm{d}z\\ &+ \int_{D\times\Sigma_{K+1}} \psi(z)\sigma^{(1)}_{lk}(y)\partial_{y_l}\overline{u}^{(1)}(z)\partial_{y_k}S(y,z)\, \mathrm{d}y\, \mathrm{d}z\\ &+ \int_{D\times D} \sigma^{(2)}_{lk}(z)\partial_{z_l}\overline{u}^{(2)}(z)\sigma^{(1)}_{nm}(y)\partial_{y_n}\overline{u}^{(1)}(z)\partial_{z_k}\partial_{y_m}S(y,z)\, \mathrm{d}y\, \mathrm{d}z = 0. \end{aligned} \tag{4.9}$$

Hence

$$\left\langle \psi, \left(\mathcal{N}^{\Sigma_{K+1}}_{\sigma^{(1)}} - \mathcal{N}^{\Sigma_{K+1}}_{\sigma^{(2)}}\right)\ \psi \right\rangle = \int_E \left(\sigma^{(2)}(x) - \sigma^{(1)}(x)\right) \nabla_x u^{(1)}(x) \cdot \nabla_x u^{(2)}(x)\, \mathrm{d}x = 0, \tag{4.10}$$

which concludes the proof of the claim. □

From $\mathcal{N}^{\Sigma_{K+1}}_{\sigma^{(1)}} = \mathcal{N}^{\Sigma_{K+1}}_{\sigma^{(2)}}$ and by lemma 3.6 we obtain

$$\sigma^{(1)}(x) = \sigma^{(2)}(x), \qquad \text{for any} \quad x \in \Sigma_{K+1},$$

hence

$$\sigma^{(1)}(x) = \sigma^{(2)}(x), \qquad \text{for any} \quad x \in D_{K+1},$$

which concludes the proof.

Example 4.2. Let $v = (v', v_n) \in \mathbb{R}^n_+$ be an arbitrary point (note that $v_n > 0$). Consider the matrix

$$M = \left(\begin{array}{c|c} I_{(n-1)} & v' \\ \hline 0'^T & v_n \end{array}\right),$$

where we understand

$$v' = \begin{pmatrix} v_1 \\ \vdots \\ v_{n-1} \end{pmatrix}$$

and $0'$ denotes the column null $(n-1)$-vector. M is a linear transformation of $\mathbb{R}^n_+$ into itself which fixes the boundary Π_n. Let us form

$$\sigma = \frac{QQ^T}{\det Q},$$

where $Q = M^{-1}$. σ is the push-forward of the isotropic homogeneous conductivity I through the change of coordinates $x = M\xi$. In this case

$$\begin{aligned} g = M^T M &= \left(\begin{array}{c|c} I_{(n-1)} & 0' \\ \hline v'^T & v_n \end{array}\right) \left(\begin{array}{c|c} I_{(n-1)} & v' \\ \hline 0'^T & v_n \end{array}\right) \\ &= \left(\begin{array}{c|c} I_{(n-1)} & v' \\ \hline v'^T & |v'|^2 + v_n^2 \end{array}\right). \end{aligned}$$

Hence

$$g_{(n-1)} = I_{(n-1)},$$

for any choice of $v \in \mathbb{R}^n_+$.

In other words, the whole family of anisotropic conductivities

$$\sigma = \frac{QQ^T}{\det Q} = v_n \left(\begin{array}{c|c} I_{(n-1)} + \frac{1}{v_n^2} v' v'^T & -\frac{1}{v_n^2} v' \\ \hline -\frac{1}{v_n^2} v'^T & \frac{1}{v_n^2} \end{array}\right)$$

is such that

$$N_\sigma(x', y') = N_I(x', y') \qquad \text{for all} \quad x', y' \in \Pi_n.$$

That is, any such σ is indistinguishable from the identity I when the corresponding N-D map (or D-N map) on Π_n is given.

5. Conclusions

The problem of uniquely determining anisotropic conductivities by local measurements has been studied in this paper. The conductivity is assumed to be piecewise constant on a domain $\Omega \subset \mathbb{R}^n$, $n \geqslant 3$, with unknown constant, matrix-valued functions in each subdomain of a given (known) partition of Ω. Such partition needs to satisfy the additional property of having contiguous subdomains joined by *curved* smooth surfaces. The measurements are collected on some open set $\Sigma \subset \partial\Omega$ that contains a *curved* portion of a surface. Under such assumptions it was shown, in theorem 2.1, that a local boundary map (localized on Σ) uniquely determines the conductivity, also in the interior. This is, to our knowledge, the first result of uniqueness of anisotropic conductivities from local measurements in dimension $n \geqslant 3$. The additional requirement of taking the measurement on a *curved* portion of the boundary appears to be a necessary assumption: this was elucidated in example 4.2 where, in the case when Ω is the half-space and measurements are taken on the flat hyperplane $\Pi_n = \{x_n = 0\}$, uniqueness is no longer obtained.

It is anticipated that the present paper will serve as a first step towards future work that will lead to a better understanding of what the obstructions to uniqueness (in the case of anisotropy) are and that further developments of the result presented here will follow from it. After the appearance of the preprint of the present paper, a more general result in this direction has been written by the same authors together with Sincich. In this subsequent manuscript [Al-dH-G-S 2], the problem of determining simultaneously the piecewise constant conductivity matrix and the (unknown) interfaces defining the partition of Ω, is studied and a result of uniqueness is presented in a similar setting. At the same time, in the corresponding inverse problem in elastostatics, Cârstea, Honda and Nakamura [C-H-N] have also obtained uniqueness from a local boundary map of a piecewise constant anisotropic elasticity tensor. In fact, in [C-H-N] the partition is also allowed to be unknown, provided it is formed by subanalytic sets, [B-M].

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