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Semiclassical inverse spectral problem for seismic surface waves in isotropic media: part I. Love waves

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Abstract
We analyze the inverse spectral problem on the half line associated with elastic surface waves. Here, we focus on Love waves. Under certain generic conditions, we establish uniqueness and present a reconstruction scheme for the S-wavespeed with multiple wells from the semiclassical spectrum of these waves.

Keywords: inverse spectral problem, semiclassical analysis, elastic surface waves

(Some figures may appear in colour only in the online journal)

1. Introduction

We analyze the inverse spectral problem on the half line associated with elastic surface waves. Here, we focus on Love waves. In a companion paper we present the corresponding inverse problem for Rayleigh waves. Surface waves have played a key role in revealing Earth's structure from the shallow near-surface to several hundred kilometers deep into the mantle, depending on the frequencies and data acquisition configurations considered.

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1.1. Seismology

The inverse spectral problem for surface waves fits in the seismological framework of surface-wave tomography. Surface-wave tomography has a long history. Since pioneering work on inference from the dispersion of surface waves half a century ago [5, 13, 15, 18, 24, 26, 32, 34, 36], surface wave tomography based on dispersion of waveforms from earthquake data has played an important role in studies of the structure of the Earth's crust and upper mantle on both regional and global scales [4, 14, 19, 20, 22, 23, 25, 27, 28, 33, 35, 37, 40].

In order to avoid the effects of scattering due to complex crustal structure, these studies focused on the analysis, measurement, and inversion of surface wave dispersion at relatively low frequencies (that is, 4–20 mHz, or periods between 50 to 250 s) at which the fundamental modes sense mantle structure to 200–300 km depth and higher modes reach across the upper mantle and transition zone to some 660 km depth. Most methods assume some form of (WKB) asymptotic and path-average approximation [10] in line with our semiclassical point of view.

More than a decade ago, Campillo and his collaborators discovered that cross correlation of ambient noise yields Green's function for surface waves [12, 30, 31]. This enabled the possibility to extend the applicability of surface-wave tomography not only to any area where seismic sensors can be placed, but also to short-path measurements and frequencies at which the data are most sensitive to shallow depths. Crustal studies based on ambient noise tomography are typically conducted in the period band of 5–40 s, but shorter period surface waves (~ 1 s, using station spacing of ~ 20 km or less) have been used to investigate shallow crustal or even near surface shear-wave speed variations [17, 21, 29, 38–40].

1.2. Semiclassical analysis perspective

In a separate contribution [11], we presented the semiclassical analysis of surface waves. Such an analysis leads to a geometric-spectral description of the propagation of these waves [1, 36]. This semiclassical analysis is built on the work of Colin de Verdière [7, 8]. *The main contribution of this paper is the construction of the Bohr–Sommerfeld quantization for Love waves.* Colin de Verdière also considered the inverse spectral problem of scalar surface waves allowing wavespeed profiles that contain a well [9]. His result does not account for the Neumann boundary condition at the surface, although a reflection principle could be invoked, but his methodology directly applies once the Bohr–Sommerfeld quantization is obtained. The reflection principle does not apply to general elastic surface waves and the remedy is presented in this paper. In the process, we show that with the Neumann boundary condition at the surface, in fact, ambiguities arising in the recovery of the S-wave speed on the line (that is, without this boundary condition) can be resolved.

We study the elastic wave equation in $X = \mathbb{R}^2 \times (-\infty, 0]$. In coordinates,

$$(x, z), \quad x = (x_1, x_2) \in \mathbb{R}^2,\ z \in \mathbb{R}^- = (-\infty, 0],$$

we consider solutions, $u = (u_1, u_2, u_3)$, satisfying the Neumann boundary condition at $\partial X = \{z = 0\}$, to the system

$$\begin{aligned} \partial_t^2 u_i + M_{il} u_l &= 0, \\ u(t=0, x, z) = 0,\ \ \partial_t u(t=0, x, z) &= h(x, z), \\ \frac{c_{i3kl}}{\rho} \partial_k u_l(t, x, z=0) &= 0, \end{aligned} \tag{1}$$

where

$$M_{il} = -\frac{\partial}{\partial z}\frac{c_{i33l}(x,z)}{\rho(x,z)}\frac{\partial}{\partial z} - \sum_{j,k=1}^{2}\frac{c_{ijkl}(x,z)}{\rho(x,z)}\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_k} - \sum_{j=1}^{2}\frac{\partial}{\partial x_j}\frac{c_{ij3l}(x,z)}{\rho(x,z)}\frac{\partial}{\partial z}$$
$$- \sum_{k=1}^{2}\frac{c_{i3kl}(x,z)}{\rho(x,z)}\frac{\partial}{\partial z}\frac{\partial}{\partial x_k} - \sum_{k=1}^{2}\left(\frac{\partial}{\partial z}\frac{c_{i3kl}(x,z)}{\rho(x,z)}\right)\frac{\partial}{\partial x_k}$$
$$- \sum_{j,k=1}^{2}\left(\frac{\partial}{\partial x_j}\frac{c_{ijkl}(x,z)}{\rho(x,z)}\right)\frac{\partial}{\partial x_k}.$$

Here, the stiffness tensor, c_{ijkl}, and density, ρ, are smooth and obey the following scaling: introducing $Z = \frac{z}{\epsilon}$,

$$\frac{c_{ijkl}}{\rho}(x,z) = C_{ijkl}\left(x, \frac{z}{\epsilon}\right), \quad \epsilon \in (0, \epsilon_0];$$

$$C_{ijkl}(x,Z) = C_{ijkl}(x,Z_I) = C^I_{ijkl}(x), \quad Z \leqslant Z_I < 0.$$

As discussed in [11], with the Neumann boundary condition, surface waves travel along the surface $z = 0$.

The remainder of the paper is organized as follows. In section 2, we give the formulation of the inverse problems as an inverse spectral problem on the half line. In section 3, we treat the simple case of recovery of a monotonic profile of wave speed. In section 4, we discuss the relevant Bohr–Sommerfeld quantization, which is the corner stone in the study of the inverse spectral problem. In section 5, we give the reconstruction scheme under generic assumptions.

2. Semiclassical description of Love waves

2.1. Surface wave equation, trace and the data

For the convenience of the readers, we briefly summarize the semiclassical description of elastic surface waves. The operator M can be viewed as a semiclassical pseudodifferential operator in (x_1, x_2) with small parameter ϵ. The leading-order (operator-valued) symbol associated with M_{il} is given by

$$H_{0,il}(x,\xi) = -\frac{\partial}{\partial Z}C_{i33l}(x,Z)\frac{\partial}{\partial Z} - \mathrm{i}\sum_{j=1}^{2}C_{ij3l}(x,Z)\xi_j\frac{\partial}{\partial Z}$$
$$- \mathrm{i}\sum_{k=1}^{2}C_{i3kl}(x,Z)\frac{\partial}{\partial Z}\xi_k - \mathrm{i}\sum_{k=1}^{2}\left(\frac{\partial}{\partial Z}C_{i3kl}(x,Z)\right)\xi_k$$
$$+ \sum_{j,k=1}^{2}C_{ijkl}(x,Z)\xi_j\xi_k. \tag{2}$$

Here we use the standard quantization of the symbol [42, section 4.1]. We view $H_0(x,\xi)$ as ordinary differential operators in Z, with domain

$$\mathcal{D} = \left\{ v \in H^2(\mathbb{R}^-) \,\Big|\, \sum_{l=1}^{3} \left(C_{i33l}(x,0)\frac{\partial v_l}{\partial Z}(0) + \mathrm{i}\sum_{k=1}^{2} C_{i3kl}\xi_k v_l(0) \right) = 0 \right\}.$$

For an isotropic medium,

$$C_{ijkl} = \hat{\lambda}\delta_{ij}\delta_{kl} + \hat{\mu}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}),$$

where $\hat{\lambda} = \frac{\lambda}{\rho}$ and $\hat{\mu} = \frac{\mu}{\rho}$. The S-wavespeed, c_S, is then $c_S = \sqrt{\hat{\mu}}$. The decoupling of Love and Rayleigh waves is observed in practice, and explained in [11]. We denote

$$P(\xi) = \begin{pmatrix} |\xi|^{-1}\xi_2 & |\xi|^{-1}\xi_1 & 0 \\ -|\xi|^{-1}\xi_1 & |\xi|^{-1}\xi_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then

$$P(\xi)^{-1}H_0(x,\xi)P(\xi) = \begin{pmatrix} H_0^{\mathrm{L}}(x,\xi) & \\ & H_0^{\mathrm{R}}(x,\xi) \end{pmatrix},$$

where

$$H_0^{\mathrm{L}}(x,\xi)\varphi_1 = -\frac{\partial}{\partial Z}\hat{\mu}\frac{\partial \varphi_1}{\partial Z} + \hat{\mu}\,|\xi|^2\varphi_1 \tag{3}$$

supplemented with boundary condition

$$\frac{\partial \varphi_1}{\partial Z}(0) = 0,$$

for Love waves. We will consider only the Love waves in this paper.

We assume that $\Lambda_\alpha(x,\xi)$ is an eigenvalue of $H_0(x,\xi)$ with eigenfunction $\Phi_{\alpha,0}(Z,x,\xi)$. By [11, theorem 2.1], we have

$$H_0^{\mathrm{L}} \circ \Phi_{\alpha,0} = \Phi_{\alpha,0} \circ \Lambda_\alpha + \mathcal{O}(\epsilon). \tag{4}$$

We define

$$J_{\alpha,\epsilon}(Z,x,\xi) = \frac{1}{\sqrt{\epsilon}}\Phi_{\alpha,0}(Z,x,\xi). \tag{5}$$

Microlocally (in x), we can construct approximate solutions of the system (1) with initial values

$$h(x,\epsilon Z) = \sum_{\alpha=1}^{\mathfrak{M}} J_{\alpha,\epsilon}(Z,x,\epsilon D_x)W_{\alpha,\epsilon}(x,Z),$$

representing surface waves. We assume that all eigenvalues $\Lambda_1 < \cdots < \Lambda_\alpha < \cdots < \Lambda_{\mathfrak{M}}$ are eigenvalues of the operator given in (3). We let $W_{\alpha,\epsilon}$ solve the initial value problems (up to leading order)

$$[\epsilon^2\partial_t^2 + \Lambda_\alpha(x,D_x)]W_{\alpha,\epsilon}(t,x,Z) = 0, \tag{6}$$

$$W_{\alpha,\epsilon}(0,x,Z) = 0, \quad \partial_t W_{\alpha,\epsilon}(0,x,Z) = J_{\alpha,\epsilon}W_\alpha(x,Z), \tag{7}$$

$\alpha = 1, \ldots, \mathfrak{M}$. We let $\mathcal{G}_0(Z, x, t, Z', \xi; \epsilon)$ denote the approximate Green's function (microlocalized in x), up to leading order, for Love waves. We may write [11]

$$\mathcal{G}_0(Z, x, t, Z', \xi; \epsilon)$$
$$= \sum_{\alpha=1}^{\mathfrak{M}} J_{\alpha,\epsilon}(Z, x, \xi) \left(\frac{\mathrm{i}}{2} \mathcal{G}_{\alpha,+,0}(x, t, \xi, \epsilon) - \frac{\mathrm{i}}{2} \mathcal{G}_{\alpha,-,0}(x, t, \xi, \epsilon) \right) \Lambda_\alpha^{-1/2}(x, \xi) \, J_{\alpha,\epsilon}(Z', x, \xi), \quad (8)$$

where $\mathcal{G}_{\alpha,\pm,0}$ are Green's functions for half wave equations associated with (6) and (7). We have the trace

$$\int_{\mathbb{R}^-} \widehat{\epsilon \partial_t \mathcal{G}_0}(Z, x, \omega, Z, \xi; \epsilon) \, \mathrm{d}\epsilon Z = \sum_{\alpha=1}^{\mathfrak{M}} \delta(\omega^2 - \Lambda_\alpha(x, \xi)) \Lambda_\alpha^{1/2}(x, \xi) + \mathcal{O}(\epsilon^{-1})$$

from which we can extract the eigenvalues Λ_α, $\alpha = 1, 2, \ldots, \mathfrak{M}$ as functions of ξ. We use these to recover the profile of c_S^2.

In practice, these eigenvalues are obtained from surface-wave tomography and to ensure that all eigenvalues are observed, measurements of surface-waveforms should be taken in boreholes. Most seismic observations are made at or near Earth's surface, but modern networks increasingly include borehole sensors indeed. For example, the Hi-net seismographic network in Japan[6] includes more than 750 sensors located in > 100 m deep boreholes and permanent sites of USArray[7] include sensors placed around 100 m depth.

2.2. Semiclassical spectrum

From here on, we only consider the operator $H_0^{\mathrm{L}}(x, \xi)$ for Love waves. We suppress the dependence on x, and introduce $h = |\xi|^{-1}$ as another semiclassical parameter. Within this setting, we also change the notation from $\frac{\partial}{\partial Z}$ to $\frac{\mathrm{d}}{\mathrm{d}Z}$. We arrive at the operator

$$L_h = -h^2 \frac{\mathrm{d}}{\mathrm{d}Z} \left(\hat{\mu}(Z) \frac{\mathrm{d}}{\mathrm{d}Z} \right) + \hat{\mu}(Z)$$

with Neumann boundary condition at $Z = 0$. The assumption on the stiffness tensor gives us the following assumption on $\hat{\mu}$:

Assumption 2.1. The (unknown) function $\hat{\mu}$ satisfies $\hat{\mu}(Z) = \hat{\mu}(Z_I)$ for all $Z \leqslant Z_I$ and

$$0 < \hat{\mu}(0) = E_0 = \inf_{Z \leqslant 0} \hat{\mu}(Z) < \hat{\mu}_{\mathrm{I}} = \sup_{Z \leqslant 0} \hat{\mu}(Z) = \hat{\mu}(Z_I).$$

The assumption that $\hat{\mu}$ attains its mininum at the boundary, and its maximum in some deep zone, is realistic in practice.

We first observe that the spectrum of L_h is divided in two parts,

$$\sigma(L_h) = \sigma_{pp}(L_h) \cup \sigma_{ac}(L_h),$$

where the point spectrum $\sigma_{pp}(L_h)$ consists of a finite number of eigenvalues in $(E_0, \hat{\mu}_{\mathrm{I}})$ and the continuous spectrum $\sigma_{ac}(L_h) = [\hat{\mu}_{\mathrm{I}}, \infty)$. We write $\lambda_\alpha = h^2 \Lambda_\alpha$. Since this is a one-dimensional problem, the eigenvalues are simple and satisfy

$$E_0 < \lambda_1(h) < \lambda_2(h) < \cdots < \lambda_{\mathfrak{M}}(h) < \hat{\mu}_{\mathrm{I}};$$

[6] http://hinet.bosai.go.jp
[7] http://usarray.org

the number of eigenvalues, $\mathfrak{M}$ increases as h decreases.

We will study how to reconstruct the profile $\hat{\mu}$ using only the asymptotic behavior of $\lambda_\alpha(h)$ in h. To this end, we introduce the semiclassical spectrum as in [9].

Definition 2.1. For given E with $E_0 < E \leqslant \hat{\mu}_{\mathrm{I}}$ and positive real number N, a sequence $\mu_\alpha(h)$, $\alpha = 1, 2, \ldots$ is a semiclassical spectrum of L_h mod $o(h^N)$ in $(-\infty, E)$ if, for all $\lambda_\alpha(h) < E$,

$$\lambda_\alpha(h) = \mu_\alpha(h) + o(h^N)$$

uniformly on every compact subset K of $(-\infty, E)$.

3. Reconstruction of a monotonic profile

In this section, we give a reconstruction scheme for the simple situation where the profile $\hat{\mu}$ is monotonic. First it is well known that

Lemma 3.1. *The first eigenvalue of L_h satisfies* $\lim_{h\to 0} \lambda_1(h) = E_0$.

Similar to theorem 3 in [7], we have

Theorem 3.1. *Assume that $\hat{\mu}$ is decreasing in $[Z_I, 0]$ (then assumption 2.1 is satisfied). Then the asymptotics of the discrete spectra $\lambda_j(h), 1 \leqslant j \leqslant \mathfrak{M}_j$ as $h \to 0$ determine the function $\hat{\mu}$.*

Before giving the proof, we recall the Abel transform and its inverse. We introduce

$$\mathcal{A}g(E) = \int_{E_0}^{E} \sqrt{E-u}\, g(u)\ \mathrm{d}u.$$

Then

$$\frac{\mathrm{d}}{\mathrm{d}E}\mathcal{A}g(E) = \frac{1}{2}Tg(E), \quad Tg(E) = \int_{E_0}^{E} \frac{g(u)}{\sqrt{E-u}}\mathrm{d}u,$$

where Tg denotes the Abel transform of g. By the inversion formula for the Abel transform,

$$\frac{\mathrm{d}}{\mathrm{d}E}T^2 g(E) = \pi g(E),$$

we get

$$\left(\frac{4}{\pi}\frac{\mathrm{d}^2}{\mathrm{d}E^2}\mathcal{A}\frac{\mathrm{d}}{\mathrm{d}E}\mathcal{A}\right) g(E) = g(E). \tag{9}$$

Proof. First, we note that $E_0 = \hat{\mu}(0)$ is determined by the first semiclassical eigenvalue $\lambda_1(h)$ by lemma 3.1. Then, we invoke Weyl's law. For $E < \hat{\mu}_{\mathrm{I}}$, let $N(h, E) = \#\{\lambda_j(h) \leqslant E\}$, where $\lambda_j(h)$ is an eigenvalue for L_h. Then [11]

$$N(h, E) = \frac{1}{2\pi h}\left[\mathrm{area}(\{(Z,\zeta) : \hat{\mu}(Z)(1+\zeta^2) \leqslant E\}) + o(1)\right]. \tag{10}$$

Thus, from the leading order asymptotic behavior (in h) of $\lambda_j(h)$ we can recover

$$\mathrm{Area}(\{(Z,\zeta) : \hat{\mu}(Z)(1+\zeta^2) \leqslant E\}) = 2\tilde{S}_0^1(E), \quad \tilde{S}_0^1(E) = \int_{f(E)}^{0} \sqrt{\frac{E-\hat{\mu}}{\hat{\mu}}}\mathrm{d}Z,$$

with $\hat{\mu}(f(E)) = E$. We change variable of integration, $Z = f(u)$, with

$$\frac{\mathrm{d}}{\mathrm{d}Z}\hat{\mu}(Z)\bigg|_{Z=f(u)} = \frac{1}{f'(u)} \tag{11}$$

and get

$$\tilde{S}_0^1(E) = \mathcal{A}g(E), \quad g(u) = \frac{f'(u)}{\sqrt{u}}.$$

Applying (9) above, we recover g, that is,

$$f'(E) = \left(\frac{4}{\pi}\sqrt{E}\frac{\mathrm{d}^2}{\mathrm{d}E^2}\mathcal{A}\frac{\mathrm{d}}{\mathrm{d}E}\right)\tilde{S}_0^1(E), \quad E_0 < E < \hat{\mu}_{\mathrm{I}}.$$

Then

$$f(E) = \int_{E_0}^{E} f'(u)\,\mathrm{d}u,$$

using that $f(E_0) = 0$ and knowledge of $E_0 = \hat{\mu}(0)$ from the first eigenvalue (lemma 3.1), from which we recover $\hat{\mu}$ by the inverse function theorem. ☐

4. Bohr–Sommerfeld quantization

The Bohr–Sommerfeld rules give a quantization for the semiclassical spectrum [6]. We will derive these rules making use of the WKB–Maslov ansatz for the eigenfunctions. We obtain an alternative proof to the one given in [8, 9], which enables to explicitly incorporate Neumann boundary conditions at the surface. It opens the way for studying inverse problems also for Rayleigh waves; these will be investigated in the subsequent paper.

We construct WKB solutions of the form

$$u_h(Z) = C\,\exp\left[\frac{1}{h}\sum_{j=0}^{\infty} h^j \mathcal{S}_j(Z)\right] \tag{12}$$

that satisfy

$$-h^2\hat{\mu}(Z)u_h''(Z) - h^2\hat{\mu}'(Z)u_h'(Z) + \hat{\mu}(Z)u_h(Z) = Eu_h(Z). \tag{13}$$

We will follow various calculations from [3] in the following analysis.

4.1. Half well

We first consider the eigenvalue problem (13) on the half line $\mathbb{R}^-$, with Neumann boundary condition at $Z = 0$. We further assume that there exists a unique Z_E such that $\hat{\mu}(Z_E) = E$. For exposition of the construction, we change the variable $Z \to Z_E - Z$ such that $\hat{\mu}(0) = E$ and Z_E is the boundary point. Furthermore, we assume that $\hat{\mu}(Z) - E > 0$ for $Z > 0$ and $\hat{\mu}(Z) - E < 0$ for $Z_E < Z < 0$. The original domain $(-\infty, 0]$ changes to $[Z_E, \infty)$. We divide the domain $[Z_E, \infty)$ into three regions: region I$(Z > 0)$, region II ($|Z|$ is small) and region III$(Z_E \leqslant Z < 0)$. We will construct WKB solutions in each region and glue them together.

First, we construct the WKB solution, $u_{\mathrm{I}}(Z)$, in region I. We substitute solutions of the form (12), collect terms of equal orders in h, and arrive at an infinite family of equations which may be solved recursively. The $\mathcal{O}(h^0)$ terms give the eikonal equation for $\mathcal{S}_0$,

$$\hat{\mu}(Z)(1-(\mathcal{S}_0'(Z))^2)=E.$$

We select the solution

$$\mathcal{S}_0(Z)=-\int_0^Z\sqrt{\frac{\hat{\mu}-E}{\hat{\mu}}}\,\mathrm{d}Z'. \tag{14}$$

Then the $\mathcal{O}(h)$ term yields

$$\hat{\mu}\mathcal{S}_0''+2\hat{\mu}\mathcal{S}_0'\mathcal{S}_1'+\hat{\mu}'\mathcal{S}_0'=0,$$

which implies that

$$\mathcal{S}_1'=-\frac{1}{2}(\log(\hat{\mu}\mathcal{S}_0'))'=-\frac{1}{4}(\log[\hat{\mu}(\hat{\mu}-E)])';$$

we select the solution

$$\mathcal{S}_1=-\frac{1}{4}\,\log[\hat{\mu}(\hat{\mu}-E)]. \tag{15}$$

The lower order terms give us a sequence of equations,

$$2\hat{\mu}\mathcal{S}_0'\mathcal{S}_j'+(\hat{\mu}\mathcal{S}_{j-1}')'+\hat{\mu}\sum_{k=1}^{j-1}\mathcal{S}_{j-k}'\mathcal{S}_k'=0,\quad j\geqslant 2.$$

We write down the explicit form of $\mathcal{S}_2$ for later use

$$\mathcal{S}_2(\delta,Z)=\int_\delta^Z\left[\frac{(E\hat{\mu}'-2\hat{\mu}\hat{\mu}')^2}{32\hat{\mu}^{3/2}(\hat{\mu}-E)^{5/2}}+\frac{-E^2\hat{\mu}''+3E\hat{\mu}\hat{\mu}''-2\hat{\mu}^2\hat{\mu}''+E(\hat{\mu}')^2}{8(\hat{\mu}-E)^{5/2}\hat{\mu}^{1/2}}\right]\mathrm{d}Z', \tag{16}$$

up to a constant difference; here, δ is any small fixed positive constant. Upon integrating by parts, we obtain

$$\begin{aligned}\mathcal{S}_2(\delta,Z)=&-\frac{(3E+2\hat{\mu})\hat{\mu}'}{48\hat{\mu}^{1/2}(\hat{\mu}-E)^{3/2}}-\frac{\hat{\mu}'}{24(\hat{\mu}-E)^{1/2}\hat{\mu}^{1/2}}\\&+\int_\delta^Z\left[-\frac{(\hat{\mu}')^2}{24\hat{\mu}^{3/2}(\hat{\mu}-E)^{1/2}}+\frac{(7E-8\hat{\mu})\hat{\mu}''}{48\hat{\mu}^{1/2}(\hat{\mu}-E)^{3/2}}\right]\mathrm{d}Z'.\end{aligned} \tag{17}$$

Next, we consider region II containing the turning point. When $|Z|$ is small, we expand

$$\hat{\mu}(Z)-E=a_1Z+a_2Z^2+a_3Z^3+\cdots.$$

Here, $a_1>0$. We write $u_{\mathrm{II}}(Z)=\hat{\mu}^{-1/2}(Z)\,v_{\mathrm{II}}(Z)=(E+a_1Z+a_2Z^2+a_3Z^3+\cdots)^{-1/2}v_{\mathrm{II}}(Z)$. Then we obtain

$$-h^2\frac{\mathrm{d}}{\mathrm{d}Z}\left(\hat{\mu}\frac{\mathrm{d}u_{\mathrm{II}}}{\mathrm{d}Z}\right)=-h^2\frac{\mathrm{d}}{\mathrm{d}Z}\left(\hat{\mu}\frac{\mathrm{d}}{\mathrm{d}Z}\hat{\mu}^{-1/2}(Z)v_{\mathrm{II}}(Z)\right)=-h^2\hat{\mu}^{1/2}v_{\mathrm{II}}''+h^2(\hat{\mu}^{1/2})''v_{\mathrm{II}}.$$

Thus, by (13), we have the following equation for v_{II}:

$$h^2 v_{\mathrm{II}}'' = \left(1 - E\hat{\mu}^{-1} + h^2 \frac{(\hat{\mu}^{1/2})''}{\hat{\mu}^{1/2}}\right) v_{\mathrm{II}}. \tag{18}$$

We further employ the simple asymptotic expansion

$$1 - E\hat{\mu}^{-1}(Z) = b_1 Z + b_2 Z^2 + \cdots,$$

where $b_1 = \frac{a_1}{E}$ and $b_2 = \frac{a_2 E - a_1^2}{E^2}$. Temporarily, we introduce the scaling $Z = h^{2/3} b_1^{-1/3} Y$. With abuse of notation for v_{II}, (18) gives

$$-h^2 h^{-4/3} b_1^{2/3} \frac{\mathrm{d}^2 v_{\mathrm{II}}}{\mathrm{d}Y^2} = (b_1 h^{2/3} b_1^{-1/3} Y + b_2 h^{4/3} b_1^{-2/3} Y^2 + \cdots) \, v_{\mathrm{II}},$$

which can be simplified to

$$\frac{\mathrm{d}^2 v_{\mathrm{II}}}{\mathrm{d}Y^2} \sim (Y + h^{2/3} b_1^{-4/3} b_2 Y^2) \, v_{\mathrm{II}}, \tag{19}$$

keeping the second-order approximation. We then seek an approximate solution of the form

$$v_{\mathrm{II}}(Y) \sim (1 + \alpha_1 h^{2/3} Y) \, \mathrm{Ai}(Y + \beta_1 h^{2/3} Y^2),$$

where Ai is the Airy function and α_1 and β_1 are constants to be determined. By tedious calculations, we find that

$$\begin{aligned} \frac{\mathrm{d}^2 v_{\mathrm{II}}}{\mathrm{d}Y^2} \sim D \Big[& \alpha_1 h^{2/3} \mathrm{Ai}'(Y + \beta_1 h^{2/3} Y^2) + \alpha_1 h^{2/3} (1 + 2\beta_1 h^{2/3} Y) \, \mathrm{Ai}'(Y + \beta_1 h^{2/3} Y^2) \\ & + (1 + \alpha_1 h^{2/3} Y)(1 + 2\beta_1 h^{2/3} Y)^2 \mathrm{Ai}''(Y + \beta_1 h^{2/3} Y^2) \\ & + (1 + \alpha_1 h^{2/3} Y) \, 2\beta_1 h^{2/3} \mathrm{Ai}'(Y + \beta_1 h^{2/3} Y^2) \Big]. \end{aligned}$$

Comparing this equation with differential equation (19), and using the property for Airy functions,

$$\mathrm{Ai}''(Y + \beta_1 h^{2/3} Y^2) = (Y + \beta_1 h^{2/3} Y^2) \, \mathrm{Ai}(Y + \beta_1 h^{2/3} Y^2),$$

we must have

$$\alpha_1 + \beta_1 = 0,$$

and

$$5\beta_1 = b_1^{-4/3} b_2.$$

Hence, undoing the scaling and returning to the original (depth) coordinate, we have

$$v_{\mathrm{II}}(Z) \sim D \left(1 - \frac{b_2}{5b_1} Z\right) \mathrm{Ai} \left[b_1^{1/3} h^{-2/3} \left(Z + \frac{b_2 Z^2}{5b_1}\right)\right].$$

Or equivalently, we write

$$u_{\mathrm{II}}(Z) \sim D\left(E^{-1/2} - \frac{1}{2}E^{-3/2}a_1 Z\right)\left(1 - \frac{a_2E - a_1^2}{5Ea_1}Z\right)\mathrm{Ai}\left[\left(\frac{a_1}{E}\right)^{1/3} h^{-2/3}\left(Z + \frac{a_2E - a_1^2}{5Ea_1}Z^2\right)\right]. \tag{20}$$

We examine $u_{\mathrm{I}}(Z)$ for small Z. We make the following approximations:

$$[\hat{\mu}(Z)(\hat{\mu}(Z) - E)]^{-1/4} \sim Z^{-1/4}(Ea_1)^{-1/4}\left(1 - \frac{1}{4}\frac{Ea_2 + a_1^2}{Ea_1}Z\right),$$

$$\int_0^Z \sqrt{\frac{\hat{\mu} - E}{\hat{\mu}}}\,\mathrm{d}Z' \sim \frac{2}{3}Z^{3/2}\left(\frac{a_1}{E}\right)^{1/2} + \frac{Ea_2 - a_1^2}{5Ea_1}\left(\frac{a_1}{E}\right)^{1/2}Z^{5/2},$$

$$\mathcal{S}_2(\delta, Z) \sim -\frac{5}{48}E^{1/2}a_1^{-1/2}Z^{-3/2} - \frac{E^{1/2}a_2a_1^{-3/2}}{12}\delta^{-1/2}.$$

In the asymptotic expansion of $\mathcal{S}_2$, we neglect terms $\mathcal{O}(Z^{-1/2})$, which is justified because $hZ^{-1/2}$ is small (compared to $h\delta^{-1/2}$, $hZ^{-3/2}$) in the limit $h \to 0$. Substituting these formulas into u_{I} gives

$$\begin{aligned} u_{\mathrm{I}} \sim{}& CZ^{-1/4}(Ea_1)^{-1/4}\left(1 - \frac{1}{4}\frac{Ea_2 + a_1^2}{Ea_1}Z\right)\exp\left[-\frac{2}{3h}Z^{3/2}\left(\frac{a_1}{E}\right)^{1/2}\right. \\ &- \frac{1}{5h}\frac{Ea_2 - a_1^2}{Ea_1}\left(\frac{a_1}{E}\right)^{1/2}Z^{5/2} - \frac{h}{48}E^{1/2}a_1^{-5/2}Z^{-3/2} \\ &\left. - \frac{h}{12}a^{-3/2}(2a_1 - E)E^{-1/2}Z^{-3/2} - \frac{hE^{1/2}a_2a_1^{-3/2}}{12}\mu^{-1/2}\right]. \end{aligned}$$

In order to glue u_{I} and u_{II}, we revisit asymptotic of $u_{\mathrm{II}}(Z)$ (20). We employ the asymptotic behavior of the Airy function $\mathrm{Ai}(s)$ for large positive s,

$$\mathrm{Ai}(s) \sim \frac{1}{2\sqrt{\pi}}s^{-1/4}\left(1 - \frac{5}{48}s^{-3/2}\right)\exp\left[-\frac{2}{3}s^{3/2}\right]$$

to obtain

$$\begin{aligned} u_{\mathrm{II}}(Z) \sim{}& D\frac{1}{2\sqrt{\pi}}\left(\frac{a_1}{E}\right)^{-1/12}h^{1/6}Z^{-1/4}E^{-1/2}\left(1 - \frac{Ea_2 + a_1^2}{4Ea_1}Z\right)\left(1 - \frac{5}{48}hZ^{-3/2}\left(\frac{a_1}{E}\right)^{-1/2}\right) \\ &\times \exp\left[-\frac{2}{3}\left(\frac{a_1}{E}\right)^{1/2}h^{-1}Z^{3/2}\left(1 + \frac{3}{2}\left(\frac{a_2E - a_1^2}{5Ea_1}\right)Z\right)\right]. \end{aligned}$$

Uniformly asymptotically matching u_{I} and u_{II} then leads to the relation of the constants C and D:

$$C = \frac{D}{2\sqrt{\pi}}h^{1/6}\exp\left[\frac{hE^{1/2}a_2a_1^{-3/2}}{12}\mu^{-1/2}\right]a_1^{1/6}E^{-1/3}. \tag{21}$$

In region III, we construct the (oscillatory) WKB solution,

$$u_{\mathrm{III}}(Z) \sim F[(E-\hat{\mu})\hat{\mu}]^{-1/4} \exp\left[\frac{\mathrm{i}}{h}\mathcal{S}_0(Z) + \mathrm{i}h\mathcal{S}_2(\mu, Z)\right] \\ + G[(E-\hat{\mu})\hat{\mu}]^{-1/4} \exp\left[-\frac{\mathrm{i}}{h}\mathcal{S}_0(Z) - \mathrm{i}h\mathcal{S}_2(\mu, Z)\right], \quad Z \to 0^-,\ h \to 0^+, \tag{22}$$

where

$$\mathcal{S}_0(Z) = \int_Z^0 \sqrt{\frac{E-\hat{\mu}}{\hat{\mu}}}\,\mathrm{d}Z' \tag{23}$$

and

$$\begin{aligned}\mathcal{S}_2(\delta, Z) &= -\int_Z^{-\delta} \left[\frac{(E\hat{\mu}' - 2\hat{\mu}\hat{\mu}')^2}{32\hat{\mu}^{3/2}(E-\hat{\mu})^{5/2}} + \frac{-E^2\hat{\mu}'' + 3E\hat{\mu}\hat{\mu}'' - 2\hat{\mu}^2\hat{\mu}'' + E(\hat{\mu}')^2}{8(E-\hat{\mu})^{5/2}\hat{\mu}^{1/2}}\right]\mathrm{d}Z' \\ &= \frac{(3E+2\hat{\mu})\hat{\mu}'}{48\hat{\mu}^{1/2}(E-\hat{\mu})^{3/2}} + \frac{\hat{\mu}'}{24(E-\hat{\mu})^{1/2}\hat{\mu}^{1/2}} \\ &\quad + \int_Z^{-\delta}\left[\frac{(\hat{\mu}')^2}{24\hat{\mu}^{3/2}(E-\hat{\mu})^{1/2}} - \frac{(7E-8\hat{\mu})\hat{\mu}''}{48\hat{\mu}^{1/2}(E-\hat{\mu})^{3/2}}\right]\mathrm{d}Z'.\end{aligned} \tag{24}$$

Next, we uniformly asymptotically match u_{II} and u_{III}. To this end, we consider the asymptotic behavior of $\mathrm{Ai}(s)$ for large negative s,

$$\mathrm{Ai}(s) \sim \frac{1}{\sqrt{\pi}}(-s)^{-1/4} \sin\left[\frac{2}{3}(-s)^{3/2} + \frac{\pi}{4}\right],$$

and obtain

$$u_{\mathrm{II}}(Z) \sim D\frac{1}{\sqrt{\pi}}\left(\frac{a_1}{E}\right)^{-1/12} h^{1/6}(-Z)^{-1/4}E^{-1/2}\left(1 - \frac{Ea_2 + a_1^2}{4Ea_1}Z\right) \\ \times \sin\left[\frac{2}{3}\left(\frac{a_1}{E}\right)^{1/2} h^{-1}(-Z)^{3/2}\left(1 + \frac{3}{2}\left(\frac{a_2E - a_1^2}{5Ea_1}\right)Z\right) + \frac{\pi}{4}\right], \quad Z \to 0^-,\ h \to 0^+.$$

Matching requires that $u_{\mathrm{III}}(Z)$ has the form

$$u_{\mathrm{III}}(Z) \sim \frac{D}{\sqrt{\pi}}\left(\frac{a_1}{E}\right)^{-1/12} h^{1/6}E^{-1/2}[(E-\hat{\mu})\hat{\mu}]^{-1/4} \\ \times \sin\left[\frac{1}{h}\mathcal{S}_0(Z) + \frac{\pi}{4} + h\mathcal{S}_2(\delta, Z) - \frac{hE^{1/2}a_2a_1^{-3/2}}{12}\delta^{-1/2}\right], \quad Z \to 0^-,\ h \to 0^+.$$

Thus,

$$F = \frac{D}{2\sqrt{\pi}}\left(\frac{a_1}{E}\right)^{-1/12} h^{1/6}E^{-1/2} \exp\left[\frac{\mathrm{i}\pi}{4} - \mathrm{i}\frac{hE^{1/2}a_2a_1^{-3/2}\delta^{-1/2}}{12}\right], \tag{25}$$

$$G = -\frac{D}{2\sqrt{\pi}}\left(\frac{a_1}{E}\right)^{-1/12} h^{1/6}E^{-1/2} \exp\left[-\frac{\mathrm{i}\pi}{4} + \mathrm{i}\frac{hE^{1/2}a_2a_1^{-3/2}\delta^{-1/2}}{12}\right]. \tag{26}$$

This completes the construction of WKB solutions.

The **Neumann boundary condition** pertains to region III, is applied at $Z = Z_E$ in the shifted coordinate and yields the Bohr–Sommerfeld rule. It takes the implicit form

$$\cot\left[\frac{1}{h}\mathcal{S}_0(Z_E) + \frac{\pi}{4} + h\mathcal{S}_2(\delta, Z_E) - \frac{hE^{1/2}a_2a_1^{-3/2}}{24}\delta^{-1/2}\right] = \mathfrak{F}(h, E),$$

$$\mathfrak{F}(h, E) = \left.\frac{h(E-2\hat{\mu})\hat{\mu}'}{4(E-\hat{\mu})\hat{\mu}\left(-\sqrt{\frac{E-\hat{\mu}}{\hat{\mu}}} + h^2\mathcal{S}_2'\right)}\right|_{Z=Z_E}. \tag{27}$$

We carry out an asymptotic expansion of $\cot^{-1}(\mathfrak{F}(h, E))$ in the small h limit

$$\cot^{-1}(\mathfrak{F}(h, E)) = \frac{\pi}{2} + h\mathfrak{F}_1(E) + \mathcal{O}(h^2),$$

where

$$\mathfrak{F}_1(E) = \left.\frac{(E-2\hat{\mu})\hat{\mu}'}{4(E-\hat{\mu})^{3/2}\hat{\mu}^{1/2}}\right|_{Z=Z_E}.$$

We undo the shift and return to the original (depth) coordinate. We consider, again, a function f such that $\hat{\mu}(f(E)) = E$ when $Z_E = f(E)$. Substituting (23) and (24), (27) takes the form

$$\begin{aligned}
&\frac{1}{h}\int_{f(E)}^{0}\sqrt{\frac{E-\hat{\mu}}{\hat{\mu}}}\,\mathrm{d}Z + \frac{\pi}{4} + \frac{(3E+2\hat{\mu}(0))\hat{\mu}'(0)}{48\hat{\mu}^{1/2}(E-\hat{\mu}(0))^{3/2}} + \frac{\hat{\mu}'(0)}{24(E-\hat{\mu}(0))^{1/2}\hat{\mu}^{1/2}(0)}\\
&\quad + \int_{f(E)+\delta}^{0}\left[\frac{(\hat{\mu}')^2}{24\hat{\mu}^{3/2}(E-\hat{\mu})^{1/2}} - \frac{(7E-8\hat{\mu})\hat{\mu}''}{48\hat{\mu}^{1/2}(E-\hat{\mu})^{3/2}}\right]\mathrm{d}Z - \frac{hE^{1/2}a_2a_1^{-3/2}}{24}\delta^{-1/2}\\
&= \left(\alpha - \frac{1}{2}\right)\pi + h\mathfrak{F}_1(E), \quad \alpha = 1, 2, \ldots,
\end{aligned}$$

where

$$\mathfrak{F}_1(E) = \frac{(-E+2\hat{\mu}(0))\hat{\mu}'(0)}{4(E-\hat{\mu}(0))^{3/2}\hat{\mu}^{1/2}(0)}.$$

By letting $\delta \downarrow 0$, using that

$$\hat{\mu}(f(E)+\delta) - E \sim -a_1\delta + a_2\delta^2,$$

where $a_1 > 0$, and that

$$\frac{E^{1/2}a_2a_1^{-3/2}}{24}\delta^{-1/2} \sim -\frac{E\hat{\mu}''(f(E))}{12\sqrt{\hat{\mu}(f(E))(E-\hat{\mu}(f(E)-\delta))}}\frac{1}{\hat{\mu}'(E)},$$

we obtain the quantization rule,

$$\frac{1}{h}\frac{1}{4}\widetilde{S}_0(E) + \frac{\pi}{4} + h\frac{1}{4}\widetilde{S}_2(E) = \left(\alpha - \frac{1}{2}\right)\pi + \mathcal{O}(h^2),$$

where

$$\widetilde{S}_0(E) = 4\int_{f(E)}^{0} \sqrt{\frac{E-\hat{\mu}}{\hat{\mu}}}\, \mathrm{d}Z \tag{28}$$

and

$$\begin{aligned}\frac{1}{4}\widetilde{S}_2(E) = & \frac{(3E+2\hat{\mu}(0))\hat{\mu}'(0)}{48\hat{\mu}^{1/2}(0)(E-\hat{\mu}(0))^{3/2}} + \frac{\hat{\mu}'(0)}{24(E-\hat{\mu}(0))^{1/2}\hat{\mu}^{1/2}(0)} \\ & -\frac{1}{24}\frac{\mathrm{d}}{\mathrm{d}E}\widetilde{J}(E) - \frac{1}{8}\widetilde{K}(E) - \mathfrak{F}_1(E),\end{aligned} \tag{29}$$

in which

$$\widetilde{J}(E) = \int_{f(E)}^{0} \left(E\hat{\mu}'' - \frac{2(E-\hat{\mu})}{\hat{\mu}}(\hat{\mu}')^2\right) \frac{\mathrm{d}Z}{\sqrt{\hat{\mu}(E-\hat{\mu})}}, \tag{30}$$

$$\widetilde{K}(E) = \int_{f(E)}^{0} \hat{\mu}'' \frac{\mathrm{d}Z}{\sqrt{\hat{\mu}(E-\hat{\mu})}}. \tag{31}$$

Remark 4.1. The above quantization rule suggest that $\lambda_1 = E_0 + \mathcal{O}(h^{2/3})$ under assumption 2.1, since the first eigenvalue is (semiclassically) associated with the half well. This would give us an improved version of lemma 3.1. If $\hat{\mu}'(0) = 0$, then the same quantization rule would lead to $\lambda_1 = E_0 + \mathcal{O}(h)$.

4.2. Full well

Here, we consider the eigenvalue problem on the entire real line. We assume that there are two simple turning points, at $Z = f_-(E)$ and at $Z = f_+(E)$; that is, $\hat{\mu} < E$ on $(f_-(E), f_+(E))$, and $\hat{\mu} > E$ on $(-\infty, f_-(E))$ and $(f_+(E), +\infty)$. Clearly, $\hat{\mu}(f_-(E)) = \hat{\mu}(f_+(E)) = E$. Similar to the half-well case, now, we construct WKB solutions in the different regions and match them in the neighborhoods of the two turning points $f_-(E)$ and $f_+(E)$. We let $a_{1,-}, a_{2,-}$ and $a_{1,+}, a_{2,+}$ be the expansion coefficients of $\hat{\mu} - E$ in the neighborhoods of $f_-(E)$ and $f_+(E)$, respectively. We now have

$$\begin{aligned}& \lim_{\delta\downarrow 0}\int_{f_-(E)+\delta}^{f_+(E)-\delta} -\frac{7E-8\hat{\mu}''}{48\hat{\mu}^{1/2}(E-\hat{\mu})^{3/2}}\,\mathrm{d}Z - \frac{E^{1/2}a_{2,-}a_{1,-}^{-3/2}}{12}\delta^{-1/2} - \frac{E^{1/2}a_{2,+}a_{1,+}^{-3/2}}{12}\delta^{-1/2} \\ & = \lim_{\delta\downarrow 0}\int_{f_-(E)+\delta}^{f_+(E)-\delta}\left(-\frac{\hat{\mu}''}{24\hat{\mu}^{1/2}(E-\hat{\mu})^{1/2}} + \frac{E\hat{\mu}''}{48\hat{\mu}^{1/2}(E-\hat{\mu})^{3/2}}\right)\mathrm{d}Z \\ & \quad + \frac{E\hat{\mu}''(f_-(E))}{24\sqrt{\hat{\mu}(f_-(E))(E-\hat{\mu}(f_-(E)+\delta))}}\frac{1}{\hat{\mu}'(f_-(E))} \\ & \quad - \frac{E\hat{\mu}''(f_+(E))}{24\sqrt{\hat{\mu}(f_+(E))(E-\hat{\mu}(f_+(E)-\delta))}}\frac{1}{\hat{\mu}'(f_+(E))} - \int_{f_-(E)}^{f_+(E)}\frac{\hat{\mu}''}{8\hat{\mu}^{1/2}(E-\hat{\mu})^{1/2}}\,\mathrm{d}Z \\ & \quad + \int_{f_-(E)}^{f_+(E)}\frac{(\hat{\mu}')^2}{24\hat{\mu}^{3/2}(E-\hat{\mu})^{1/2}}\,\mathrm{d}Z = -\frac{1}{24}\frac{\mathrm{d}}{\mathrm{d}E}J(E) - \frac{1}{8}K(E),\end{aligned}$$

where

$$J(E) = \int_{f_-(E)}^{f_+(E)} \left(E\hat{\mu}'' - \frac{2(E-\hat{\mu})}{\hat{\mu}}(\hat{\mu}')^2\right)\frac{\mathrm{d}Z}{\sqrt{\hat{\mu}(E-\hat{\mu})}}, \tag{32}$$

$$K(E) = \int_{f_-(E)}^{f_+(E)} \hat{\mu}'' \frac{\mathrm{d}Z}{\sqrt{\hat{\mu}(E-\hat{\mu})}}. \tag{33}$$

That is, we arrive at the quantization rule

$$\frac{1}{h}\frac{1}{2}S_0(E) + h\frac{1}{2}S_2(E) \sim \left(\alpha - \frac{1}{2}\right)\pi,$$

where

$$S_0(E) = \frac{1}{2}\int_{f_-(E)}^{f_+(E)} \sqrt{\frac{E-\hat{\mu}}{\hat{\mu}}}\,\mathrm{d}Z \tag{34}$$

and

$$S_2(E) = -\frac{1}{12}\frac{\mathrm{d}}{\mathrm{d}E}J(E) - \frac{1}{4}K(E). \tag{35}$$

We note that the above form has also been derived in [9] using the method introduced in [6].

4.3. Multiple wells

In the case of multiple wells we invoke

Assumption 4.1. There is a $Z^* < 0$ such that $\hat{\mu}'(Z^*) = 0$, $\hat{\mu}''(Z^*) < 0$ and $\hat{\mu}'(Z) < 0$ for $Z \in (Z^*, 0)$.

Assumption 4.2. The function $\hat{\mu}(Z)$ has non-degenerate critical values at a finite set

$$\{Z_1, Z_2, \ldots, Z_M\}$$

in $(Z_I, 0)$ and all critical points are non-degenerate extrema. None of the critical values of $\hat{\mu}(Z)$ are equal, that is, $\hat{\mu}(Z_j) \neq \hat{\mu}(Z_k)$ if $j \neq k$.

We label the critical values of $\hat{\mu}(Z)$ as $E_1 < \cdots < E_M < \hat{\mu}_\mathrm{I}$ and the corresponding critical points by $Z_1, \ldots, Z_M$. We use the fact that $\hat{\mu}(0) = \inf_{Z\leqslant 0}\hat{\mu}(Z)$ and denote $Z_0 = 0$ and $E_0 = \hat{\mu}(Z_0)$.

We define a well of order k as a connected component of $\{Z \in (Z_I, 0) : \hat{\mu}(Z) < E_k\}$ that is not connected to the boundary, $Z = 0$. We refer to the connected component connected to the boundary as a half well of order k. We denote $J_k = (E_{k-1}, E_k)$, $k = 1, 2, 3, \ldots$. and let N_k ($\leqslant k$) be the number of wells of order k (see figure 1 top). The set $\{Z \in (Z_I, 0) : \hat{\mu}(Z) < E_k\}$ consists of N_k wells $W_j^k(E)$, $j = 1, 2, \ldots, N_k$, and one half well $\widetilde{W}^k(E)$ such that

$$(\cup_{j=1}^{N_k} W_j^k(E)) \cup \widetilde{W}^k(E) \subset [Z_I, 0).$$

The half well $\widetilde{W}^k(E)$ is connected to the boundary $Z = 0$.

Similar to proposition 10.1 in [9], we can divide the semiclassical spectrum of L_h in J_k into $N_k + 1$ parts, where each part is associated with a single well or half-well. The result is summarized in the following proposition.

Proposition 4.1. *The semiclassical spectrum of L_h mod $o(h^{5/2})$ in J_k is the union of $N_k + 1$ spectra: $\cup_{j=1}^{N_k}\Sigma_j^k(h) \cup \widetilde{\Sigma}^k(h)$. Here, $\Sigma_j^k(h)$ is the semiclassical spectrum associated to well W_j^k, and $\widetilde{\Sigma}^k(h)$ is the semiclassical spectrum for half well $\widetilde{W}^k$.*

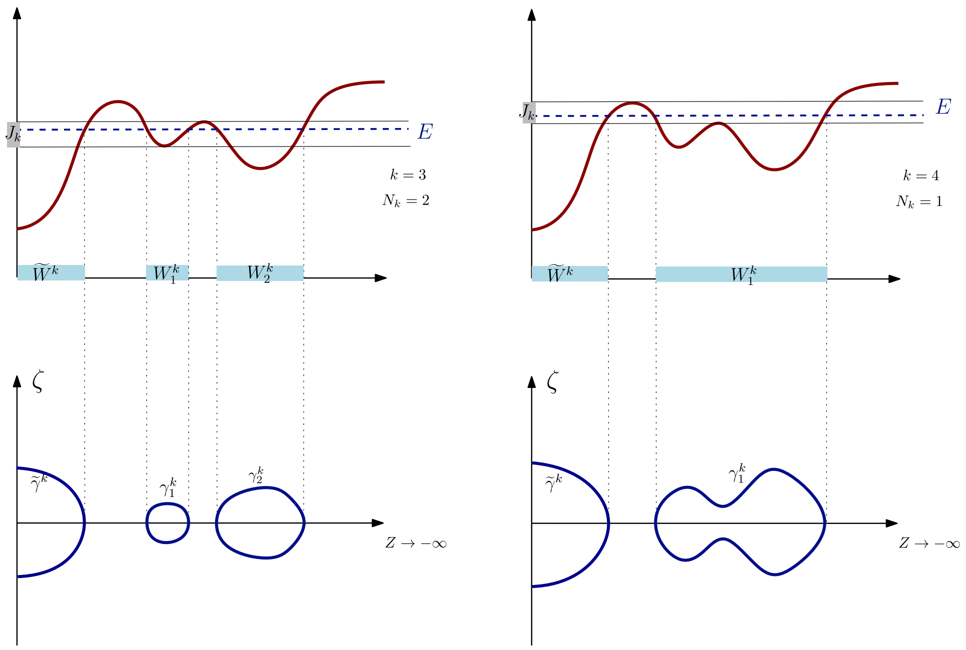

Figure 1. Wells of different orders and periodic trajectories.

The above separation of semiclassical spectra comes from the fact that the eigenfunctions are $\mathcal{O}(h^\infty)$ outside the wells, and is related to the exponentially small 'tunneling' effects [16, 41]. For a full well the quantization rule for $\Sigma_j^k(h)$ is the same as the whole real line case, and for the half well the quantization rule for $\widetilde{\Sigma}^k(h)$ is the same as the half line case. We refer further to [3] for more details. Therefore, we have Bohr–Sommerfeld rules for separated wells, that is,

$$\Sigma_j^k(h) = \{\mu_\alpha(h) : E_{k-1} < \mu_\alpha(h) < E_k \text{ and } S^{k,j}(\mu_\alpha(h)) = 2\pi h\alpha\}, \tag{36}$$

where $S^{k,j} = S^{k,j}(E) : (E_{k-1}, E_k) \to \mathbb{R}$ admits the asymptotics in h

$$S^{k,j}(E) = S_0^{k,j}(E) + h\pi + h^2 S_2^{k,j}(E) + \cdots$$

and

$$\widetilde{\Sigma}^k(h) = \{\nu_\alpha(h) : E_{k-1} < \nu_\alpha(h) < E_k \text{ and } \widetilde{S}^k(\nu_\alpha(h)) = 2\pi h\alpha\}, \tag{37}$$

where $\widetilde{S}^k = \widetilde{S}^k(E) : (E_{k-1}, E_k) \to \mathbb{R}$ admits the asymptotics

$$\widetilde{S}^k(E) = \frac{1}{2}\tilde{S}_0^k(E) + \frac{3}{2}h\pi + \frac{1}{2}h^2\widetilde{S}_2^k(E) + \cdots .$$

The form of $S^{k,j}$ is similar to the one given in (34) and (35) and the form of $\widetilde{S}^k$ is similar to the one given in (28)–(31). We will give more details below.

For alternative representations of $S^{k,j}$ and $\widetilde{S}^k$, we introduce the classical Hamiltonian $p_0(Z,\zeta) = \hat{\mu}(Z)(1+\zeta^2)$. For any k, $p_0^{-1}(J_k)$ is a union of N_k topological annuli A_j^k and a

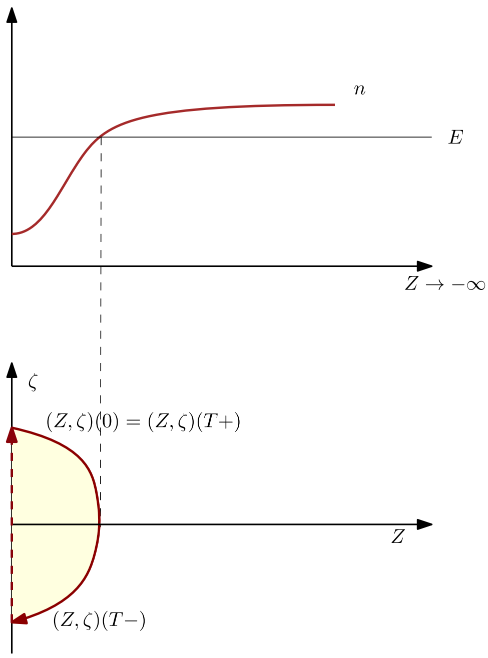

Figure 2. Behavior of a half trajectory.

half annulus $\widetilde{A}^k$. The map $p_0 : A_j^k \to J_k$ is a fibration whose fibers $p_0^{-1}(E) \cap A_j^k$ are topological circles $\gamma_j^k(E)$ that are periodic trajectories of classical dynamics (illustrated in figure 1 bottom). The map $p_0 : \widetilde{A}^k \to J_k$ is a topological half circle $\widetilde{\gamma}^k(E)$. If $E \in J_k$ then $p_0^{-1}(E) = (\cup_{j=1}^{N_k} \gamma_j^k(E)) \cup \widetilde{\gamma}^k(E)$. The corresponding classical periods are

$$T_j^k(E) = \int_{\gamma_j^k(E)} |\mathrm{d}t|.$$

We let t be the parametrization of $\gamma_j^k(E)$ by time evolution in

$$\frac{\mathrm{d}Z}{\mathrm{d}t} = \partial_\zeta p_0, \quad \frac{\mathrm{d}\zeta}{\mathrm{d}t} = -\partial_Z p_0 \tag{38}$$

for a realized energy level E.

For the half well $\widetilde{W}_k$, (Z, ζ) follows a periodic (half) trajectory as shown in figure 2. After one (half-) period T, the trajectory reaches the boundary $Z(T) = 0$, and encounters a perfect reflection, so that

$$\zeta(T+) = -\zeta(T-) = \sqrt{\frac{E - \hat{\mu}(0)}{\hat{\mu}(0)}},$$

and then continues following the Hamilton system (38).

4.3.1. Wells separated from the boundary. For a well W_j^k separated from the boundary, the associated semiclassical spectrum mod $o(h^{5/2})$ follows from (36) and (32)–(35). We have

$$S^{k,j}(E) = S_0^{k,j}(E) + h\pi + h^2 S_2^{k,j}(E), \tag{39}$$

where

$$S_0^{k,j}(E) = \int_{\gamma_j^k(E)} \zeta \, \mathrm{d}Z = \mathrm{area}\{(Z,\zeta) : p_0(Z,\zeta) \leqslant E,\ Z \in W_j^k\} \tag{40}$$

and

$$S_2^{k,j}(E) = -\frac{1}{12}\frac{\mathrm{d}}{\mathrm{d}E}\int_{\gamma_j^k(E)} \left(E\hat{\mu}'' - 2\frac{(E-\hat{\mu})}{\hat{\mu}}(\hat{\mu}')^2 \right) |\mathrm{d}t| - \frac{1}{4}\int_{\gamma_j^k(E)} \hat{\mu}''|\mathrm{d}t|. \tag{41}$$

The explicit forms of $S_0^{k,j}$ and $S_2^{k,j}$ are equivalent to those given in (34) and (33). Here, the integration over $(f_-(E), f_+(E))$, $E \in [E_{k-1}, E_k]$, in the Z coordinate has been changed into integration along the periodic trajectory γ. One can get the same results by using the method in [6, 9]. From (40) it is immediate that

$$(S_0^{k,j})'(E) = T_j^k(E). \tag{42}$$

4.3.2. Half well connected to the boundary. For the half well $\widetilde{W}^k$ connected to the boundary, we have, mod $\mathcal{O}(h^2)$,

$$\widetilde{S}^k(E) = \frac{1}{2}\widetilde{S}_0^k(E) + h\frac{3}{2}\pi, \tag{43}$$

where

$$\widetilde{S}_0^k(E) = 2\int_{\tilde{\gamma}^k(E)} \zeta \, \mathrm{d}Z. \tag{44}$$

The explicit form of $\widetilde{S}_0^k$ is equivalent to the one given in (28). Here, the integration over $(f(E), 0)$, $E \in [E_{k-1}, E_k]$, in the Z coordinate has been changed into integration along the (half) periodic trajectory $\tilde{\gamma}$. As before, it follows that

$$\frac{1}{2}(\widetilde{S}_0^k)'(E) = \frac{1}{2}\widetilde{T}^k(E). \tag{45}$$

The explicit form of $\widetilde{S}_2^k$ will not be needed in the following and hence we omit it.We note that $S_0^{k,j}$ and $\widetilde{S}_0^k$ depend only on periodic trajectories.

Remark 4.2. In the further analysis of the inverse problem, the explicit form of S_2^k is only needed for the wells separated from the boundary (between two turning points) and there the formulas are exactly as in [9] (on the whole line without boundary conditions). Near the boundary (between a turning point and the boundary) the function $\hat{\mu}$ is strictly decreasing and only S_0^k or the counting function for semiclassical eigenvalues suffice to reconstruct the profile.

5. Unique recovery of $\hat{\mu}$ from the semiclassical spectrum

5.1. Trace formula

The inverse problem is addressed with a trace formula as it reflects the data.

Lemma 5.1. ([9], lemma 11.1). *Let $S : J \to \mathbb{R}$ be a smooth function with $S' > 0$. Then we have the following identity as Schwartz distributions in J, meaning that the equality holds* *when applying both sides to a test function $\phi \in C_0^\infty(J)$,*

$$\sum_{\alpha\in\mathbb{Z}} \delta(E - S^{-1}(2\pi h\alpha)) = \frac{1}{2\pi h}\sum_{m\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}mS(E)/h} S'(E). \tag{46}$$

Substituting the action in (39), (43) and the Bohr–Sommerfeld rules in (46) yields, on J_k with $\{\mu_\alpha(h)\}_\alpha = \cup_{j=1}^{N_k}\Sigma_j^k(h)$,

$$\begin{aligned}\sum_{\alpha\in\mathbb{Z}} \delta(E-\mu_\alpha(h)) &= \frac{1}{2\pi h}\sum_{j=1}^{N_k}\sum_{m\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}m(S_0^{k,j}(E)h^{-1}+\pi+hS_2^{k,j}(E)+\mathcal{O}(h^2))}((S_0^{k,j})'(E)+h^2(S_2^{k,j})'(E)+\mathcal{O}(h^3))\\ &= \frac{1}{2\pi h}\sum_{j=1}^{N_k}\sum_{m\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}mS_0^{k,j}(E)h^{-1}}\,\mathrm{e}^{\mathrm{i}m\pi}(S_0^{k,j})'(E)(1+\mathrm{i}mhS_2^{k,j}(E)+\mathcal{O}(h^2))\end{aligned}$$

and with $\{\nu_\alpha(h)\}_\alpha = \widetilde{\Sigma}^k(h)$:

$$\begin{aligned}\sum_{\alpha\in\mathbb{Z}} \delta(E-\nu_\alpha(h)) &= \frac{1}{2\pi h}\sum_{m\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}m(\frac{1}{2}\widetilde{S}_0^k(E)h^{-1}+\frac{3}{2}\pi+h\frac{1}{2}\widetilde{S}_2^k(E)+\mathcal{O}(h^2))}\left(\frac{1}{2}(\widetilde{S}_0^k)'(E)+\frac{h^2}{2}(\widetilde{S}_2^k)'(E)+\mathcal{O}(h^3)\right)\\ &= \frac{1}{2\pi h}\sum_{m\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}m\frac{1}{2}\widetilde{S}_0^k(E)h^{-1}}\,\mathrm{e}^{\mathrm{i}m\frac{3}{2}\pi}\frac{1}{2}(\widetilde{S}_0^k)'(E)\left(1+\mathrm{i}mh\frac{1}{2}\widetilde{S}_2^k(E)+\mathcal{O}(h^2)\right).\end{aligned}$$

Therefore, we have

Theorem 5.1. *Let $\{\mu_\alpha(h)\}$ be the semiclassical spectrum of $H_{0,h}$ modulo $o(h^{5/2})$. As distributions on J_k, we have*

$$\begin{aligned}\sum_{\alpha\in\mathbb{Z}} \delta(E-\mu_\alpha(h)) =& \frac{1}{2\pi h}\sum_{j=1}^{N_k}\sum_{m\in\mathbb{Z}} (-1)^m\,\mathrm{e}^{\mathrm{i}mS_0^{k,j}(E)h^{-1}}T_j^k(E)(1+\mathrm{i}mhS_2^{k,j}(E))\\ &+\frac{1}{2\pi h}\sum_{m\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}m\frac{1}{2}\widetilde{S}_0^k(E)h^{-1}}\,\mathrm{e}^{\mathrm{i}m\frac{3}{2}\pi}\widetilde{T}^k(E)\left(1+\mathrm{i}mh\frac{1}{2}\widetilde{S}_2^k(E)\right)+o(1).\end{aligned} \tag{47}$$

The direct way to obtain this trace formula is starting from (2.1), that is,

$$\int_{\mathbb{R}^-} \widehat{\epsilon\partial_t\mathcal{G}_0}(Z,x,\omega,Z,\xi;\epsilon)\,\mathrm{d}(\epsilon Z) \sim \frac{1}{2h^2}\sum_{\alpha\in\mathbb{Z}}\delta(E-\mu_\alpha(h)),$$

upon substituting $E = h^2\omega^2$. We then expand the parametrix (8) in the WKB eigenfunctions (12) from the previous section.

We will use the notation

$$Z_{m,j}^k(E) = \frac{1}{2\pi h}(-1)^m\,\mathrm{e}^{\mathrm{i}mS_0^{k,j}(E)h^{-1}}T_j^k(E)(1+\mathrm{i}mhS_2^{k,j}(E)), \quad j=1,\ldots,N_k, \tag{48}$$

$$Z_{m,N_k+1}^k(E) = \frac{1}{2\pi h}\,\mathrm{e}^{\mathrm{i}m\frac{3}{2}\pi}\,\mathrm{e}^{\mathrm{i}m\frac{1}{2}\widetilde{S}_0^k(E)h^{-1}}T_{N_k+1}^k(E)\left(1+\mathrm{i}mh\frac{1}{2}\widetilde{S}_2^k(E)\right), \tag{49}$$

$$T^k_{N_k+1}(E) := \widetilde{T}^k(E) \tag{50}$$

for $m \in \mathbb{Z}$. To further unify the notation, we write

$$S^{k,N_k+1}_{0,2}(E) := \frac{1}{2}\widetilde{S}^k_{0,2}(E).$$

The micro-support of $Z^k_{m,j}$, $j = 1, \ldots, N_k + 1$, is given by the Lagrangian submanifold

$$L^k_{m,j} = \{(E, mT^k_j(E)) : E \in J_k\}$$

of T^*J_k associated with phase function $mS^{k,j}_0(E)$.

5.2. Separation of clusters and the weak transversality condition

We observe that the singular points of the counting function, $\int_{p_0(Z,\zeta)\leqslant E} |\mathrm{d}Z\,\mathrm{d}\zeta|$, are precisely the critical values, $E_1, E_2, \ldots, E_M$, of $\hat{\mu}$ [9, lemma 11.1] and, hence, are determined using the Weyl asymptotics first. From the singularity at E_k one can extract the value of $\hat{\mu}''(Z_k)$. We then invoke

Assumption 5.1. For any $k = 1, 2, \ldots$ and any j with $1 \leqslant j < l \leqslant N_k + 1$, the classical periods (half-period if $j = N_k + 1$) $T^k_j(E)$ and $T^k_l(E)$ are weakly transverse in J_k, that is, there exists an integer N such that the Nth derivative $(T^k_j - T^k_l)^{(N)}(E)$ does not vanish.

We introduce the sets

$$B = \{E \in J_k : \exists j \neq l, \quad T^k_j(E) = T^k_l(E)\},$$

while suppressing k in the notation. By the weak transversality assumption, it follows that B is a discrete subset of J_k.

We let the distributions $D_h(E) = \sum_{\alpha\in\mathbb{Z}} \delta(E - \mu_\alpha(h))$ be given on intervals $J = J_k$ modulo $o(1)$ using (47). These distributions are determined mod $o(1)$ by the semiclassical spectra mod $o(h^{5/2})$. We denote by Z_h the finite sum defined by the right-hand side of (47) restricted to $m = 1$, that is,

$$Z_h(E) = \sum_{j=1}^{N_k+1} Z^k_{1,j}(E).$$

By analyzing the micro-support of D_h and Z_h [9, lemmas 12.2 and 12.3], we find

Lemma 5.2. *Under the weak transversality assumption, the sets B and the distributions Z_h mod $o(1)$ are determined by the distributions D_h mod $o(1)$.*

Lemma 5.3. *Assuming that the S^j's are smooth and the a_j's do not vanish, there is a unique splitting of Z_h as a sum*

$$Z_h(E) = \frac{1}{2\pi h} \sum_{j=1}^{N_k+1} (a_j(E) + hb_j(E))\mathrm{e}^{\mathrm{i}S^j(E)/h} + o(1).$$

It follows that the spectrum in J_k mod $o(h^{5/2})$ determines the actions $S^{k,j}_0(E)$, $S^{k,j}_2(E)$ and $\widetilde{S}^k_0(E)$. This provides the separation of the data for the N_k wells and the half well.

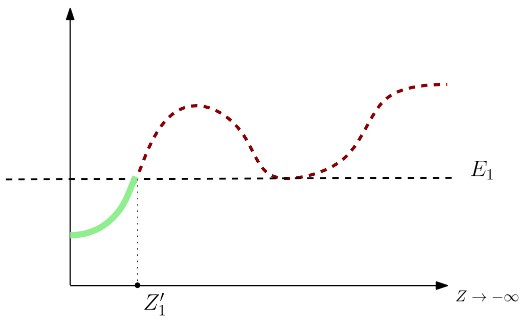

Figure 3. Reconstruction step 1 in green.

For the reconstruction of $\hat{\mu}$ from these actions, we need one more assumption

Assumption 5.2. The function $\hat{\mu}$ has a generic symmetry defect: if there exist $X_\pm$ satisfying $\hat{\mu}(X_-) = \hat{\mu}(X_+) < E$, and for all $N \in \mathbf{N}$, $\hat{\mu}^{(N)}(X_-) = (-1)^N \hat{\mu}^{(N)}(X_+)$, then $\hat{\mu}$ is globally even with respect to $\frac{1}{2}(X_+ + X_-)$ in the interval $\{Z : \hat{\mu}(Z) < E\}$.

We will carry out the reconstruction of $\hat{\mu}$ successively in intervals J_k, $k = 1, \ldots, M$ and then on the interval $[E_M, E_{M+1}]$ with $E_{M+1} = \hat{\mu}_{\mathrm{I}}$.

5.3. Reconstruction of a single well, with barrier and descreasing profile

We discuss in detail the case of one local minimum for $Z < 0$ and global minimum at $Z = 0$ ($\hat{\mu}(0) < \hat{\mu}(Z)\,\forall\, Z < 0$, $\hat{\mu}'(0) \leqslant 0$). This means that the global minimum occurs at $Z = 0$ and $E_1 = \hat{\mu}(Z_1)$ is the local minimum. Then $E_2 = \hat{\mu}(Z_2)$ is attained at $Z_2 \in (Z_1, 0)$ and $E_3 = \hat{\mu}_{\mathrm{I}}$.

Step 1. For $E \in (E_0, E_1)$, there is only one (half) well, $\widetilde{W}^1(E)$, of order 1 with $\widetilde{W}^1(E_1) = [Z_1', 0]$. Since $\hat{\mu}$ is strictly decreasing in $\widetilde{W}^1(E_1)$, we may reconstruct $\hat{\mu}$ on this interval as in section 3. This is illustrated in figure 3 in green.

Step 2. We note that Z_2 in this case is the Z^* defined above assumption 4.1. We consider $E \in (E_1, E_2)$ which corresponds to wells of order $k = 2$ with $N_k = 1$ (one connected component for $Z < 0$ separated from the boundary). The two wells are $W^{2,1}(E)$ and $\widetilde{W}^2(E)$ with $W^{2,1}(E_2) = [Z_-, Z_2]$ and $\widetilde{W}^2(E_2) = [Z_2, 0]$. Here, Z_- is the unique point in $[Z_I, Z_1]$ such that $E_2 = \hat{\mu}(Z_-)$. We are given $S_0^{2,1}$, $S_2^{2,1}$ and $\widetilde{S}_0^2$ (and $\widetilde{S}_2^2$).

We continue to reconstruct $\hat{\mu}$ from $[Z'_1, 0]$ to $[Z_2, 0]$ from $\widetilde{S}_0^2$. For the reconstruction of $\hat{\mu}$ on the interval $I = [Z_-, Z_2]$, more effort is needed. We note that, up to this point, I itself cannot be determined yet. The following theorem is a version of [9, theorem 5.1].

Theorem 5.2. *Under assumption 5.2, the function $\hat{\mu}$ is determined on I by $S_0^{2,1}$ and $S_2^{2,1}$ up to a symmetry $\hat{\mu}(Z) \to \hat{\mu}(c - Z)$, where $\frac{c}{2}$ is the midpoint of I.*

Proof. For any $E \in [E_1, E_2)$ the functions $f_\pm : [E_1, E_2) \to I$, are defined so that $W_1^2(E) = [f_-(E), f_+(E)]$. We have $\hat{\mu}'(Z) < 0$ for $Z \in (f_-(E), Z_1)$ and $\hat{\mu}'(Z) > 0$ for $Z \in (Z_1, f_+(E))$. We introduce

$$\Phi(E) = f_+'(E) - f_-'(E) \quad \text{and} \quad \Psi(E) = \frac{1}{f_+'(E)} - \frac{1}{f_-'(E)}.$$

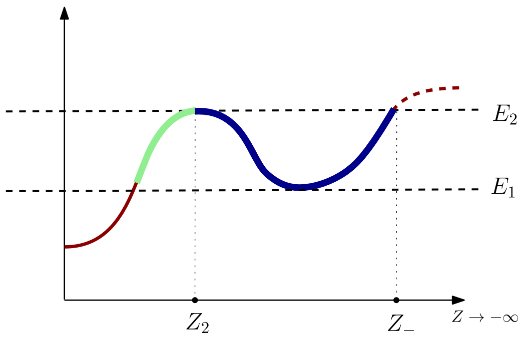

Figure 4. Reconstruction step 2, first part in green and second part in blue.

As in the proof of theorem 3.1, we have

$$(S_0^{2,1})'(E) = Tg(E), \quad Tg(E) = \int_{E_1}^{E} \frac{g(u)}{\sqrt{E-u}}\,\mathrm{d}u \quad \text{with} \quad g(u) = \frac{\Phi(u)}{\sqrt{u}}.$$

The inversion formula for the Abel transform yields $\Phi(E)$ for $E \in [E_1, E_2)$.

Concerning the recovery of Ψ, we have

$$S_2^{2,1}(E) = -\frac{1}{12}\frac{\mathrm{d}}{\mathrm{d}E}\mathcal{B}\Psi(E),$$

$$\mathcal{B}\Psi(E) = \int_{E_1}^{E} \left((7E-6u)\Psi'(u) - 2\left(\frac{E}{u}-1\right)\Psi(u)\right)\frac{\mathrm{d}u}{\sqrt{u(E-u)}},$$

which follows from (35) with (32) and (33) upon changing variable of integration, $Z = f_\pm(u)$. Thus, from $S_2^{2,1}(E)$ and the fact $\mathcal{B}\Psi(E_1) = \pi\sqrt{2E_1\hat{\mu}''(Z_1)}$, we can recover $\mathcal{B}\Psi(E)$. It can be shown that

$$\frac{\pi}{E^{3/2}}\frac{\mathrm{d}^2}{\mathrm{d}E^2}(T \circ \mathcal{B}\Psi)(E) = E^2\Psi''(E) + 4E\Psi'(E) - \Psi(E).$$

That is, we obtain a second-order inhomogeneous ordinary differential equation for Ψ on the interval $[E_1, E_2)$. This equation is supplemented with the 'initial' conditions

$$\Psi(E_1) = 0, \quad \lim_{E \downarrow E_1} \sqrt{E-E_1}\,\Psi'(E) = \sqrt{2\hat{\mu}''(Z_1)}$$

As mentioned in section 5.2, this second derivative is obtained from the limiting behavior of the counting function which coincides with $S_0^{2,1}(E)$ as $E \downarrow E_1$. We use that the period of small oscillations of the 'pendulum' associated to the local minimum of $\hat{\mu}$ at Z_1 is given by

$$(S_0^{2,1})'(E) = \int_{f_-(E)}^{f_+(E)} \frac{\mathrm{d}Z}{\sqrt{\hat{\mu}(E-\hat{\mu})}} = \pi\sqrt{\frac{2}{E_1\hat{\mu}''(Z_1)}} + o(1) \quad \text{as}\, E\downarrow E_1.$$

Thus we obtain $\Psi(E)$ for $E \in [E_1, E_2)$.

With $\pm f_\pm'(E) > 0$ for $E \in (E_1, E_2)$, we then find

$$2f_\pm' = \pm\Phi + \sigma\sqrt{\Phi^2 - 4\frac{\Phi}{\Psi}} \tag{51}$$

with

$$\sigma = \operatorname{sign}(f'_+ + f'_-) = \begin{cases} +1 & \text{if} \quad f'_+ + f'_- > 0, \\ 0 & \text{if} \quad f'_+ + f'_- = 0, \\ -1 & \text{if} \quad f'_+ + f'_- < 0. \end{cases}$$

We note that the sign is not (yet) determined, and only if the well is mirror-symmetric with respect to its vertex then $f'_+ + f'_- = 0$ and the square root in (51) vanishes. However, later, we will find the sign by a gluing argument.

By assumption 5.2, the function $\sigma = \sigma(E)$ is constant for $E \in (E_1, E_2)$. Hence, in what follows we will exchange σ with $\pm$. We have

$$f_+(E) = Z_1 + \frac{1}{2}\int_{E_1}^{E} \left(\Phi \pm \sqrt{\Phi^2 - 4\frac{\Phi}{\Psi}}\right) \mathrm{d}E,$$

$$f_-(E) = Z_1 + \frac{1}{2}\int_{E_1}^{E} \left(-\Phi \pm \sqrt{\Phi^2 - 4\frac{\Phi}{\Psi}}\right) \mathrm{d}E.$$

Since $f_+(E_2) = Z_2$ and $f_-(E_2) = Z_-$, we find that

$$Z_2 = Z_1 + \frac{1}{2}\int_{E_1}^{E_2} \left(\Phi \pm \sqrt{\Phi^2 - 4\frac{\Phi}{\Psi}}\right) \mathrm{d}E,$$

$$Z_- = Z_1 + \frac{1}{2}\int_{E_1}^{E_2} \left(-\Phi \pm \sqrt{\Phi^2 - 4\frac{\Phi}{\Psi}}\right) \mathrm{d}E.$$

Hence, the distance, $Z_2 - Z_1$, between the two critical points is recovered (modulo mirror symmetry of Z_1 with respect to $\frac{c}{2}$). Since $f_\pm$ are both monotonic on (E_1, E_2), $\hat{\mu}$ can be recovered (up to mirror symmetry) on I. $\square$

With this result, the reconstructions on $[Z'_1, 0]$ and I can be smoothly glued together, and the uncertainty in the translation of I and the 'orientation' of $\hat{\mu}$ on I are eliminated. Thus $\hat{\mu}$ is uniquely determined on the interval $[Z_-, 0]$. This is illustrated in figure 4.

Step 3. On the interval $[Z_I, Z_-]$ we may use the Weyl asymptotics again to recover $\hat{\mu}$. The counting function in the interval $[E_2, E_3]$ is obtained from $\widetilde{S}^3$ which corresponds with

$$\mathrm{Area}(\{(Z,\zeta) : \hat{\mu}(Z)(1+\zeta^2) \leqslant E\}) = A_1(E) + A_2(E),$$

where

$$A_1(E) = \mathrm{area}(\{(Z,\zeta) : \hat{\mu}(Z)(1+\zeta^2) \leqslant E, \quad Z_- \leqslant Z \leqslant 0\})$$

is already known, and

$$A_2(E) = 2\int_{f(E)}^{Z_-} \sqrt{\frac{E - \hat{\mu}}{\hat{\mu}}}\, \mathrm{d}Z,$$

$Z_I \leqslant f(E) < Z_-$ since $E_2 \leqslant E \leqslant E_3 = \hat{\mu}_{\mathrm{I}}$. Thus we may recover $\hat{\mu}$ on the interval $[Z_I, Z_-]$ where $\hat{\mu}$ is decreasing while applying theorem 3.1. Step 3 is illustrated in figure 5.

The two profiles for $\hat{\mu}$ on $[Z_I, Z_-]$ and on $[Z_-, Z_2]$ are then glued together at $Z = Z_-$ which is already known. This completes the reconstruction procedure.

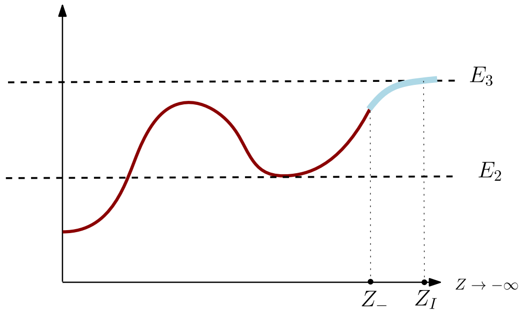

Figure 5. Reconstruction step 3 in light blue.

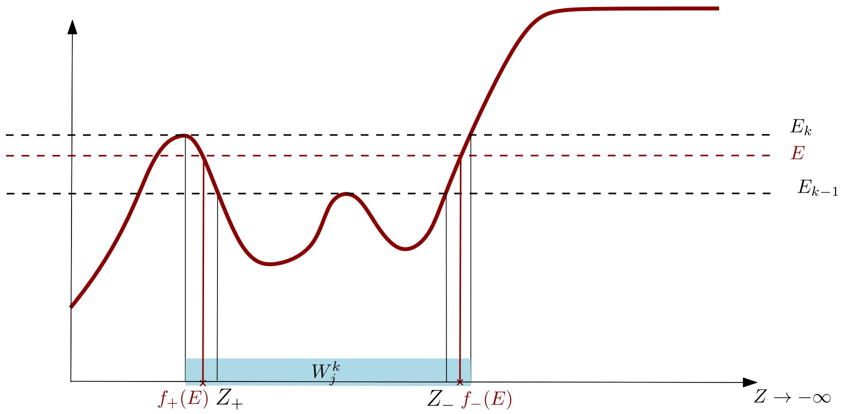

Figure 6. Illustration of $f_\pm$.

5.4. Reconstruction of multiple wells

If $\hat{\mu}$ has multiple wells, we follow an inductive procedure. First, we consider the reconstruction of the half well $\widetilde{W}^k$ of order k between E_{k-1} and E_k. We note that $\widetilde{W}^k$ must be a continuation of the half well $\widetilde{W}^{k-1}$, or be joined with some well $W_{j'}^{k-1}$ of order $k-1$. This can be done in a fashion similar to the process presented above (on $[Z_I, Z_-]$).

Secondly, we consider the reconstruction of a well, W_j^k, separated from the boundary, of order k. The well W_j^k might be a new well, and can be reconstructed as in theorem 5.2. The well, W_j^k, might also be joining two wells of order $k-1$, or extending a single well of order $k-1$. Let the profile under E_{k-1} already be recovered. The smooth joining of two wells can be carried out under assumption 5.2. We consider now functions $f_-(E)$ and $f_+(E)$ for $E \in [E_{k-1}, E_k]$ such that W_j^k is the union of three connected intervals,

$$W_j^k(E_k) = [f_-(E_k), f_-(E_{k-1})) \cup [f_-(E_{k-1}), f_+(E_{k-1})] \cup (f_+(E_{k-1}), f_+(E_k)];$$

see figure 6. The semiclassical spectrum in (E_{k-1}, E_k) up to $o(h^{5/2})$ gives the actions $S_0^{k,j}$ and $S_2^{k,j}$.

From $S_0^{k,j}$ we obtain

$$T_j^k(E) = (S_0^{k,j})'(E) = \int_{f_-(E)}^{f_+(E)} \frac{\mathrm{d}Z}{\sqrt{\hat{\mu}(E-\hat{\mu})}}$$

which signifies the periods of the trajectories of energy E. We write $Z_- = f_-(E_{k-1})$ and $Z_+ = f_+(E_{k-1})$, and decompose the interval:

$$[f_-(E), f_+(E)] = [f_-(E), Z_-) \cup [Z_-, Z_+] \cup (Z_+, f_+(E)].$$

In accordance with this decomposition,

$$T_j^k(E) = T_-(E) + T_{k-1}(E) + T_+(E),$$

where

$$\begin{aligned} T_-(E) &= \int_{f_-(E)}^{Z_-} \frac{\mathrm{d}Z}{\sqrt{\hat{\mu}(E-\hat{\mu})}}, \\ T_{k-1}(E) &= \int_{Z_-}^{Z_+} \frac{\mathrm{d}Z}{\sqrt{\hat{\mu}(E-\hat{\mu})}}, \\ T_+(E) &= \int_{Z_+}^{f_+(E)} \frac{\mathrm{d}Z}{\sqrt{\hat{\mu}(E-\hat{\mu})}}. \end{aligned}$$

We note that $T_{k-1}(E)$ is already known. In $T_\mp(E)$ we change the variable of integration, $Z = f_\mp(u)$. Using that $\hat{\mu}(f_\mp(u)) = u$, we get

$$T_\mp(E) = \mp \int_{E_{k-1}}^{E} \frac{f'_\mp(u)}{\sqrt{u(E-u)}}\,\mathrm{d}u;$$

then,

$$T_j^k(E) - T_{k-1}(E) = Tg(E), \quad Tg(E) = \int_{E_{k-1}}^{E} \frac{g(u)}{\sqrt{E-u}}\mathrm{d}u \quad \text{with} \quad g(u) = \frac{\Phi(u)}{\sqrt{u}}$$

and $\Phi(u) = f'_+(u) - f'_-(u)$ as before. Inverting this Abel transform [2], we obtain Φ on $[E_{k-1}, E_k)$.

From $S_2^{k,j}$ we obtain

$$-\frac{1}{12}\frac{\mathrm{d}}{\mathrm{d}E}J(E) - \frac{1}{4}K(E),$$

where

$$\begin{aligned} J(E) &= \int_{f_-(E)}^{f_+(E)} \left(E\hat{\mu}'' - 2\frac{(E-\hat{\mu})}{\hat{\mu}}(\hat{\mu}')^2\right) \frac{\mathrm{d}Z}{\sqrt{\hat{\mu}(E-\hat{\mu})}}, \\ K(E) &= \int_{f_-(E)}^{f_+(E)} \hat{\mu}'' \frac{\mathrm{d}Z}{\sqrt{\hat{\mu}(E-\hat{\mu})}}. \end{aligned}$$

Using that

$$\hat{\mu}(f_\pm(E)) = E, \quad \hat{\mu}'|_{Z=f_\pm(E)} = \frac{1}{f'_\pm(E)}, \quad \hat{\mu}''|_{Z=f_\pm(E)} = \left(\frac{1}{f'_\pm}\right)'(E)\frac{1}{f'_\pm(E)},$$

changing variables of integration in J and K, $Z = f_{\pm}(u)$ and introducing

$$\Psi(E) = \frac{1}{f_+'(E)} - \frac{1}{f_-'(E)},$$

we have

$$J(E) - J_{k-1}(E) = \int_{E_{k-1}}^{E} \left(E\Psi'(u) - 2\left(\frac{E}{u} - 1\right)\Psi(u)\right) \frac{\mathrm{d}u}{\sqrt{u(E-u)}},$$

$$K(E) - K_{k-1}(E) = \int_{E_{k-1}}^{E} \Psi'(u) \frac{\mathrm{d}u}{\sqrt{u(E-u)}},$$

where

$$J_{k-1}(E) = \int_{Z_-}^{Z_+} \left(E\hat{\mu}'' - 2\frac{(E-\hat{\mu})}{\hat{\mu}}(\hat{\mu}')^2 \right) \frac{\mathrm{d}Z}{\sqrt{\hat{\mu}(E-\hat{\mu})}},$$

$$K_{k-1}(E) = \int_{Z_-}^{Z_+} \hat{\mu}'' \frac{\mathrm{d}Z}{\sqrt{\hat{\mu}(E-\hat{\mu})}}$$

are already known. Thus, from $S_2^{k,j}$, we recover

$$\mathcal{B}\Psi(E) = \int_{E_{k-1}}^{E} \left((7E - 6u)\Psi'(u) - 2\left(\frac{E}{u} - 1\right)\Psi(u)\right) \frac{\mathrm{d}u}{\sqrt{u(E-u)}}.$$

Then similar to the proof of theorem 5.2, we recover Ψ on $[E_{k-1}, E_k)$ by inverting $\mathcal{B}$ through the introduction of a second-order ordinary differential equation.

From Φ and Ψ we obtain

$$2f_+' = \Phi \pm \sqrt{\Phi^2 - 4\frac{\Phi}{\Psi}}, \quad 2f_-' = -\Phi \pm \sqrt{\Phi^2 - 4\frac{\Phi}{\Psi}}$$

and then

$$f_+(E) = Z_+ + \frac{1}{2}\int_{E_{k-1}}^{E} \left(\Phi \pm \sqrt{\Phi^2 - 4\frac{\Phi}{\Psi}} \right) \mathrm{d}E,$$

$$f_-(E) = Z_- + \frac{1}{2}\int_{E_{k-1}}^{E} \left(\Phi \pm \sqrt{\Phi^2 - 4\frac{\Phi}{\Psi}} \right) \mathrm{d}E.$$

From f_- we recover $\hat{\mu}$ on the interval $[f_-(E), Z_-]$ and from f_+ we recover $\hat{\mu}$ on the interval $[Z_+, f_+(E)]$. The $\pm$ signs in $f_\pm$ are disentangled by smoothly joining the newly reconstructed pieces to the previously reconstructed part and assumption 5.2, as in previous section. Since the profile in $[Z_-, Z_+]$ can only be determined up to translation and symmetry, the determination of the profile in W_j^k is up to the same translation and symmetry.

The symmetry and translation freedom for all the wells will be gradually eliminated during the whole process. At the final step, there is a single half well connected to the boundary, and then we can reconstruct exactly the entire profile.

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