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Recovery of Material Parameters in Transversely Isotropic Media

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Abstract

In this paper we show that in anisotropic elasticity, in the particular case of transversely isotropic media, under appropriate convexity conditions, knowledge of the qSH wave travel times determines the tilt of the axis of isotropy as well as some of the elastic material parameters, and the knowledge of qP and qSV travel times conditionally determines a subset of the remaining parameters, in the sense that if some of the remaining parameters are known, the rest are determined, or if the remaining parameters satisfy a suitable relation, they are all determined, under certain non-degeneracy conditions. Furthermore, we give a precise description of the additional issues, which are a subject of ongoing work, that need to be resolved for a full treatment.

1. Introduction

In this paper we show that in anisotropic elasticity, in the particular case of transversely isotropic media, under appropriate convexity conditions, knowledge of the qSH wave travel times determines the tilt of the axis of isotropy as well as some of the elastic material parameters, and the knowledge of qP and qSV travel times conditionally determines a subset of the remaining parameters, in the sense that if some of the remaining parameters are known, the rest are determined, or if the remaining parameters satisfy a suitable relation, they are all determined, under certain non-degeneracy conditions. Furthermore, we give a precise description of the additional issues, which are a subject of ongoing work, that need to be resolved for a full treatment.

The problem addressed in this paper has one of its primary applications in seismic tomography. In Earth's interior, the presence of anisotropy has been widely recognized. In a classical (review) paper, Silver described the seismic anisotropy beneath the continents [17]. More recently, Romanowicz and Wenk [14] described anisotropy in the deep interior. The assumption of transverse isotropy with tilted symmetry axis has played a dominant role in many studies ranging from Earth's sedimentary basins, continental dynamics and subduction, deep mantle flow and inner core.

The fundamental result of this paper is that the spatially varying symmetry axis of a transversely isotropic elastic medium can be locally recovered, under certain geometric conditions. However, in the present analysis, the full recovery of elastic parameters requires some interrelationships between them. Such relationships may be best motivated by considering models that effectively generate these parameters; these then provide possible physically, mechanically or geologically based reductions of independent parameters. We briefly mention a selection of examples of modeling procedures of this kind, omitting references to a vast literature on the subject: (i) differential effective medium theories, for which we refer to NORRIS et al. [12]; effective-medium-averaging techniques to estimate the effective properties of a random sphere pack while considering contact laws for adhesive contacts, rough contacts, and smooth contacts, which were developed by DIGBY [4] and WALTON [24], which later culminated in the modeling of elastic properties of shales [9] in sedimentary basins; (ii) (sedimentary) layering-induced anisotropy in a simple calculus formulated by SCHOENBERG and MUIR [15], and more general shape preferred orientation (SPO), considered by GARNERO and MOORE [7], in its most basic form originating from the study of a deformable elastic ellipsoid in a far-field loaded matrix with different properties by ESHELBY [5,6]; and (iii) strain-induced formation of lattice preferred orientation (LPO). Indeed in Earth's upper mantle it is generally accepted that seismic anisotropy results from LPO produced by dislocation creep of olivine [25], while the mechanisms causing anisotropy in the inner core are still under debate.

In order to state the results precisely, we work in an invariant setting based on Riemannian geometry, since this enables a cleaner and conceptually clearer statement. Thus, there is a given background metric g_0, which is typically the Euclidean metric; we denote the dual metric and the dual metric function by G_0. In general, anisotropic elasticity is described by a system whose principal symbol, a tensor (matrix)-valued function on phase space, that is, the cotangent bundle, is non-scalar, that is, is not a multiple of the identity map. Also, it can be diagonalized; the eigenvalues are the speed of the various elastic waves. In isotropic elasticity, there are two kinds of waves: P and S waves, with S waves corresponding (in spatial dimension 3) to a multiplicity 2 (and P waves a simple) eigenvalue. In anisotropic elasticity, typically the S wave eigenspace is broken up, at least in most parts of the cotangent bundle. In transversely isotropic elasticity there is a preferred axis, with respect to which the principal symbol is rotationally symmetric relative to the background metric G_0 lifted to the cotangent bundle. There are three waves then, the qP waves, as well as the qSV and qSH waves, with the latter corresponding to the 'breaking up' of the S-waves. Of these, the qSH waves behave much like in *isotropic* elasticity in the sense that they are given the dual metric function of a Riemannian metric, while the qP and qSV waves have a very different character.

One common parameterization of transversely isotropic materials, see [1,22], is via the material constants a_{11}, a_{13}, a_{33}, a_{55} and a_{66}, which are functions on the underlying manifold. In addition, there is an axis of isotropy, which can be encoded by a vector field, or better yet a one form ω. The qSH 'energy function' (dual metric function) then depends on $a_{55}, a_{66} > 0$ and ω. Concretely, using orthogonal coordinates relative to the metric g_0 (with G_0 the dual metric), and aligning the axis of isotropy with the third coordinate axis, which is possible at any given point, the wave speed of the qSH waves is given by a (squared!) Riemannian dual metric

$$G = G_{qSH} = a_{66}(x)|\xi'|^2 + a_{55}(x)\xi_3^2 = a_{66}(x)G_0 + (a_{55}(x) - a_{66}(x))\xi_3^2.$$

This corresponds to a Riemannian metric

$$g = g_{qSH} = a_{66}(x)^{-1}\,|\mathrm{d}x'|^2 + a_{55}(x)^{-1}\,\mathrm{d}x_3^2 = a_{66}(x)^{-1}g_0 + (a_{55}(x)^{-1} \\ -a_{66}(x)^{-1})\,\mathrm{d}x_3^2,$$

again at the point in question. Thus, invariantly it has the form

$$g = \alpha g_0 + (\beta - \alpha)\omega \otimes \omega,$$

that is, the metric is a rank one perturbation of a conformal multiple of the background (say, Euclidean) metric, with $\alpha = a_{66}^{-1}$, $\beta = a_{55}^{-1}$ functions on the base manifold. Note that here $\beta - \alpha$ could be incorporated into ω up to a sign; this formulation keeps the sign unspecified, but then one should keep in mind that only the direction of ω matters. There is another reason to keep this form, as will be explained below. Note also that g determines the span of ω if $\beta \neq \alpha$, for the kernel of ω is well-defined (at any point in the manifold) as the 2-dimensional subspace of the tangent space restricted to which g is a constant multiple of g_0.

Now, under appropriate assumptions, for example locally near the strictly convex boundary, a Riemannian metric, g, can be recovered from its boundary distance function up to diffeomorphisms, as shown by Stefanov, Uhlmann and Vasy, recalled here in Section 2, see [20] for details. More precisely, see Section 2, the local determination indeed only uses the boundary distance function, while the global result uses the lens relation, which also keeps track of the direction of the geodesics at the two points on the boundary at which they enter and exit the domain; in many cases these are equivalent. Thus, if we know the qSH wave travel times, then in fact we know g above up to diffeomorphisms (which are the identity at the boundary). A natural question is whether this arbitrary diffeomorphism freedom is present in our problem for the qSH wave travel times.

Formal dimension counting indicates that the space of Riemannian metrics is 6 dimensional at each point, that of vector fields (or one forms) is 3 dimensional at each point, so formally the space of Riemannian metrics modulo diffeomorphisms is 3 dimensional. Now, above, α, β are arbitrary functions, and ω is arbitrary but only its direction matters, which means that the parameter space at each point is 4 dimensional. This indicates that it is unlikely that one can recover these four parameters from knowing the corresponding Riemannian metric *up to diffeomorphisms*.

On the other hand, if one assumes that ω satisfies additional conditions, this pointwise parameter space can be cut down, and the problem may become formally determined. For instance, if ω always lies in the $\mathrm{d}x_1$–$\mathrm{d}x_3$ plane, this would be the case. (Note that this includes the case when $\omega = \mathrm{d}x_3$, in which case the pointwise above form holds at least locally.)

An important property of a one-form, such as ω, is its integrability, or more precisely whether its kernel is an integrable hyperplane distribution, which means that $\operatorname{Ker}\omega$ is the tangent space of a smooth family of submanifolds, which are thus locally level sets of a function f, so ω is a smooth multiple of $\mathrm{d}f$. In this case,

$$g = \alpha g_0 + \gamma\, \mathrm{d}f \otimes \mathrm{d}f.$$

In geological terms, this corresponds to a layered material with layers given by the level sets of f. The integrability condition is natural though not globally (that is, on planetary scale). LPO is one mechanism that is consistent with this assumption while sedimentary processes, compaction and deformation would yield the condition to also hold true.

Our first theorem is

Theorem 1.1. *Consider the class of elastic problems in which* $\operatorname{Ker}\omega = \operatorname{Ker}\mathrm{d}f$ *is an integrable hyperplane distribution on a manifold with boundary M, with ω not conormal to ∂M (so level sets of f locally intersect ∂M non-degenerately) and not orthogonal to $N^*\partial M$ relative to G_0. Then, under the local, resp. global, convexity conditions for Riemannian determination (up to diffeomorphisms), stated here in Theorems 2.1 and 2.2 of Section 2, f, α, β are locally, resp. globally, determined by the qSH travel times, resp. qSH lens relations, and the labelling of the level sets of f at the boundary.*

Thus, there is *no* diffeomorphism freedom in this problem, unlike for the boundary rigidity problem in Riemannian geometry.

Since the qSH-wave speed does not depend on the remaining material parameters, a_{11}, a_{13}, a_{33}, in order to go further we need to consider qSV and qP waves. Now, at a point with coordinates g_0-orthogonal at the point and such that the isotropy axis is aligned with the $\tilde{x}_3$ axis the Hamiltonians for the other waves take the form (with $\pm$ corresponding to the choice of qP vs. qSV, and G being twice what gives the actual wave speeds)

$$\begin{aligned} G_{qP/qSV} = {} & (a_{11}+a_{55})|\tilde{\xi}'|^2 + (a_{33}+a_{55})\tilde{\xi}_3^2 \\ & \pm \sqrt{\big((a_{11}-a_{55})|\tilde{\xi}'|^2 + (a_{33}-a_{55})\tilde{\xi}_3^2\big)^2 - 4E^2|\tilde{\xi}'|^2\tilde{\xi}_3^2}, \end{aligned} \tag{1.1}$$

where

$$E^2 = (a_{11}-a_{55})(a_{33}-a_{55}) - (a_{13}+a_{55})^2,$$

see [16]. (We will make the physically natural assumption that $\max\{a_{55}, a_{66}\} < \min\{a_{11}, a_{33}\}$ throughout the paper.) If one uses another coordinate system, x_j, often chosen orthogonal at the point in question, and corresponding dual variables ξ_j, the actual wave speed is given by the corresponding change of variables. Thus, these wave speeds are no longer given by a quadratic polynomial in the fibers of the cotangent bundle, and thus are not the wave speeds of a Riemannian metric *unless*

$E = 0$, that is, E measures the departure from the Riemannian case (which is sometimes called the 'elliptic case' due to the quadratic polynomial nature). (One can say that they are the wave speeds of a co-Finsler metric due to the homogeneity with respect to dilations in the fibers of the cotangent bundle, cf. [3] for the terminology and for a detailed discussion.) Correspondingly, the Riemannian result [20], is not applicable. Nevertheless, the analysis of that paper is based on the study of a class of transforms which are microlocally weighted X-ray transforms along curves, and even these general wave speeds fall in this class, with the techniques introduced by Uhlmann and Vasy [23] being applicable.

Following [23], in this paper we work with a function on M with strictly convex level sets, and localize to super-level sets of this function. We show that the modified and localized 'normal operators' that arise from the Stefanov–Uhlmann pseudo-linearization formula, which is valid for all Hamiltonian flows and goes back to [18], are scattering pseudodifferential operators in Melrose's scattering pseudodifferential algebra [10], with the level set of the function at which we stop playing the role of the boundary. (Thus, this *artificial boundary* is the only one with analytic significance, while the original boundary of M simply constrains supports.) In this algebra, whose properties are summarized in [20, Section 3.2], there are two different (and somewhat coupled) notions of ellipticity: that of the standard principal symbol and that of the boundary principal symbol; the boundary principal symbol at infinity in the fibers of the (scattering) cotangent bundle is the same as the standard principal symbol at the boundary, explaining the coupling. The standard principal symbol corresponds to differentiable regularity, the boundary principal symbol to decay.

Now, there are three quantities we would still like to determine—a_{11}, a_{33} and E—and we have two different wave speeds, the qSV and the qP waves that we can use. While ideally one would like to determine all of these at the same time, it is at this point natural to formulate a theorem in which two of these three are regarded as known, and the third as unknown. Due to multiple points in the cotangent space potentially corresponding to the same tangent vector via the Hamilton map (a phenomenon that causes 'wave triplication'), we make an additional non-degeneracy hypothesis for the material, for which we refer to Definition 3.1.

Theorem 1.2. *Assume that the hypotheses of Theorem 1.1 hold, and that ∇f is neither parallel, nor orthogonal to the artificial boundary with respect to g_0. (This is automatic near ∂M if the convex function is a boundary defining function for ∂M, or a sufficiently small perturbation of such.) Assume moreover that the transversely isotropic material is non-degenerate relative to a convex foliation if qSV data are used below, with convexity of the foliation always understood with respect to G_{qP}, resp. G_{qSV}, if qP, resp. qSV data are used below.*

Then the modified and localized 'normal operators' arising from the Stefanov–Uhlmann *formula are in Melrose's scattering pseudodifferential operator algebra. Furthermore, the boundary principal symbol is elliptic at* finite *points of the scattering cotangent bundle for any one of E^2, a_{11}, a_{33} from the qP travel data, and for E^2 (as well as a_{11} and a_{33} if $E^2 > 0$) from the qSV travel data. Furthermore, for a_{11} from the qP-travel time data standard principal symbol ellipticity also holds.*

Note that the assumption that $\mathrm{d}f$ is not conormal to the artificial boundary, that is, ∇f is not orthogonal to it, means that the span of $\mathrm{d}f$ has a non-degenerate image in the scattering cotangent bundle; if ρ defines the artificial boundary, this is that of the scattering one-form $\rho^{-1}\,\mathrm{d}f$.

An immediate corollary, using the methods of [19,23], is

Corollary 1.1. *Suppose that we are given the qSH-travel time data so that ω, a_{55} and a_{66} are determined already, and assume that the hypotheses of Theorem 1.2 hold. Given E^2 and a_{33}, the material parameter a_{11} can be recovered from qP-travel time data.*

Motivated by the discussion in the introduction on possible parameter set reduction, by elimination we may invoke a functional relationship where a_{11} determines a_{33} and E^2. This yields an alternative to the corollary above:

Corollary 1.2. *Suppose that we are given the qSH-travel time data so that ω, a_{55} and a_{66} are determined already, and assume that the hypotheses of Theorem 1.2 hold for both the qP and qSV waves with the* same *convex foliation. Suppose also that we are given C^∞ functions F, $H: \mathbb{R} \to \mathbb{R}$ with $F' \geqq 0$ such that $a_{33} = F(a_{11})$ and $E^2 = H(a_{11})$. Then a_{11} can be recovered from the qP- and qSV-travel time data jointly.*

Finally, we show the precise nature of the obstruction to full invertibility via elliptic analysis as follows:

Theorem 1.3. *For a_{33}, E^2 from the qP or qSV travel data, as well as for E^2 and one of a_{11} and a_{33} jointly from the qP and qSV data, the standard principal symbol is* not *elliptic, rather vanishes in a non-degenerate quadratic manner along the span of $\mathrm{d}f$ at fiber infinity in the scattering cotangent bundle.*

The explanation of the lack of ellipticity is very simple: in general, for the normal operator's standard principal symbol computation at a point $\zeta \in T_x^*M$, one takes a weighted average of certain quantities evaluated at covectors for which the Hamilton vector field for the relevant wave speed is annihilated by ζ. Now, if $\zeta = \mathrm{d}f$ is in the axis direction, the tangent vectors involved in the integration correspond to covectors in the g_0-orthogonal plane, that is with a vanishing $\tilde{\xi}_3$ coordinate, and there the qP and qSV wave speeds are insensitive to a_{33}, E^2 as these appear with a prefactor $\tilde{\xi}_3^2$ above. The quadratic non-degeneracy also corresponds to this; namely, the relevant coefficient is a non-degenerate multiple of $\tilde{\xi}_3^2$.

This means that the analytic framework for this inverse problem involves double characteristics, which were studied in the paper of Guillemin and Uhlmann [8]. However, here these need to be analyzed in the context of scattering pseudodifferential operators, and the analysis must be *global* on the manifold cut out by the artificial boundary.

Of course, we would like to determine all three of the remaining parameters ideally. One may set up a system by adding a third row and using different premultipliers, as was done in [20] to treat the normal gauge, but one will certainly still have the double characteristic phenomena at the minimum.

2. Proof of Theorem 1.1

At the beginning of this section we recall the results of Stefanov, Uhlmann and Vasy [20]. The simplest result to formulate is the local boundary rigidity result in Riemannian geometry.

Theorem 2.1. ([20, Theorem 1.1]) *Suppose that* (M, g) *is an* n*-dimensional Riemannian manifold with boundary,* $n \geqq 3$*, and assume that* ∂M *is strictly convex (in the second fundamental form sense) with respect to each of the two metrics* g *and* $\hat{g}$ *at some* $p \in \partial M$*. If* $d_g|_{U\times U} = d_{\hat{g}}|_{U\times U}$*, for some neighborhood* U *of* p *in* ∂M*, then there is a neighborhood* O *of* p *in* M *and a diffeomorphism* $\psi\colon O \to \psi(O)$ *fixing* $\partial M \cap O$ *pointwise such that* $g|_O = \psi^*\hat{g}|_O$*.*

Furthermore, if the boundary is compact and everywhere strictly convex with respect to each of the two metrics g *and* $\hat{g}$ *and* $d_g|_{\partial M\times\partial M} = d_{\hat{g}}|_{\partial M\times\partial M}$*, then there is a neighborhood* O *of* ∂M *in* M *and a diffeomorphism* $\psi\colon O \to \psi(O)$ *fixing* $\partial M \cap O$ *pointwise such that* $g|_O = \psi^*\hat{g}|_O$*.*

Paper [20] also proves a global consequence of the local results, assuming that M is connected with non-trivial boundary. This global statement requires a globally defined function x with level sets which are strictly concave from the superlevel sets and which is $\geqq 0$ at a subset of ∂M; such functions necessarily exist near the boundary but not necessarily globally. (One can take for instance the negative of a boundary defining function as a local example near the boundary, though this does not localize within ∂M. See [23] for more examples.) One also has the global existence of such a function under appropriate curvature conditions; see [20] for more details and references. (As an example, for domains in non-positively curved simply connected manifolds, the distance to a point outside the domain satisfies the concavity requirements.) This theorem uses the *lens relation*, which in addition to the distance between boundary points keeps track of the directions at these points of geodesics connecting them. For simple manifolds (strictly convex boundary and the geodesic exponential map around each point is a diffeomorphism), the knowledge of the boundary distance function $d_g|_{\partial M\times\partial M}$ and of the lens relation is equivalent, see [11].

Theorem 2.2. ([20, Theorem 1.3]) *Suppose that* (M, g) *is a compact* n*-dimensional Riemannian manifold,* $n \geqq 3$*, with strictly convex boundary, and* x *is a smooth function with non-vanishing differential whose level sets are strictly concave from the superlevel sets, and* $\{\mathsf{x} \geqq 0\} \cap M \subset \partial M$*.*

Suppose also that $\hat{g}$ *is a Riemannian metric on* M *and suppose that the lens relations of* g *and* $\hat{g}$ *are the same.*

Then there exists a diffeomorphism $\psi\colon M \to M$ *fixing* ∂M *such that* $g = \psi^*\hat{g}$*.*

We start the proof of Theorem 1.1 by discussing some consequences of its integrability hypothesis.

As we already mentioned, in general $\operatorname{Ker}\omega$, thus in this case $\operatorname{Ker} \mathrm{d}f$ is well-defined, and so is its g-orthocomplement, which is the same as the g_0-orthocomplement, since if a vector W is g_0 orthogonal to an element V of

$\operatorname{Ker} \mathrm{d}f$, then $\mathrm{d}f \otimes \mathrm{d}f(V, W) = \mathrm{d}f(V)\,\mathrm{d}f(W) = 0$ shows the g-orthogonality, and conversely. Moreover, taking $y_3 = f$, one can introduce local coordinates in which ∂_{y_1}, ∂_{y_2} are orthogonal to ∂_{y_3}: one does this by defining y_j on a level set of f, and then extending them to be constant along integral curves of $\nabla^g f$. Indeed, in this case ∂_{y_1}, ∂_{y_2} are tangent to the level sets of f, for $\partial_{y_j} y_3 = 0$, $j = 1, 2$, while ∂_{y_3} is a multiple of $\nabla^g f$, which is orthogonal to the f level sets, hence to the ∂_{y_j}, $j = 1, 2$. Correspondingly, in these coordinates, the metric takes the form

$$g = \sum_{i,j=1}^{2} a_{ij}\,\mathrm{d}y_i \otimes \mathrm{d}y_j + a_{33}\,\mathrm{d}y_3 \otimes \mathrm{d}y_3.$$

Notice that, by the above remark, one has the same result if $\nabla^{g_0} f$-integral curves are used, since they are also orthogonal to the level sets of f, thus are simply reparameterizations of the $\nabla^g f$-integral curves. Furthermore, one can take *any* hypersurface transversal to $\nabla^{g_0} f$ to define y_1 and y_2 originally. Thus, if $\nabla^{g_0} f$ is not tangent to the boundary, that is, ω is not G_0-orthogonal to $N^*\partial M$, *as we have assumed*, one can use the boundary for this purpose.

Now suppose that two metrics g and $\tilde{g}$ of this form are the same up to a diffeomorphism Φ fixing the boundary, that is

$$\tilde{\alpha} g_0 + \tilde{\gamma}\,\mathrm{d}\tilde{f} \otimes \mathrm{d}\tilde{f} = \Phi^*(\alpha g_0 + \gamma\,\mathrm{d}f \otimes \mathrm{d}f).$$

Since $\operatorname{Ker} \mathrm{d}f$ is determined by g, and $\operatorname{Ker} \mathrm{d}\tilde{f}$ is determined by $\tilde{g}$, Φ preserving the metrics implies that the differential of Φ then will take $\operatorname{Ker} \mathrm{d}f$ and its g, thus g_0-orthocomplement to $\operatorname{Ker} \mathrm{d}\tilde{f}$ and its $\tilde{g}$, thus $\tilde{g}_0$-orthocomplement. Using coordinates y_j and $\tilde{y}_j$ as above this means that $D\Phi$ is block-diagonal, with the (12) and (3) blocks being non-trivial. This says that $\partial_j \Phi_3 = 0$, $j = 1, 2$, and $\partial_3 \Phi_j = 0$, $j = 1, 2$. Thus, Φ_1 and Φ_2 are independent of y_3, while Φ_3 is independent of y_1 and y_2, so if one can make the argument that (Φ_1, Φ_2) is the identity at some point of each y_3-curve, then it is so globally; moreover Φ_3 simply relabels the level sets, that is, $\tilde{y}_3 = \Phi_3(y_3)$. One can achieve this, however, by choosing y_1 and y_2 on the boundary (locally), *using that* $\nabla^{g_0} f$, $\nabla^{g_0} \tilde{f}$ *are transversal to the boundary by our assumption*, and then choosing $\tilde{y}_1$ and $\tilde{y}_2$ to be the same as y_1, y_2 there—then at the boundary, the (12) block of $D\Phi$ is the identity matrix. Moreover, *as* ∇f *is not orthogonal to* ∂M, *that is,* ω *is not conormal to* ∂M, *by our assumption*, the labelling of the level sets of f is determined by their value at the boundary (since they intersect the boundary, and they do so non-degenerately), and the same for $\tilde{f}$. Thus, in this case the diffeomorphism is the identity, and thus g is determined from the boundary distance function (locally). Since g in turn determines α, $\gamma = \beta - \alpha$, this proves Theorem 1.1.

3. Proof of Theorems 1.2 and 1.3

3.1. The Pseudolinearization Formula and Its Basic Properties

To proceed, consider the Stefanov–Uhlmann pseudolinearization formula which, as we recalled already, is valid for all Hamiltonian flows and goes back to

[18]; recall that in the isotropic setting one uses the momentum, ∂_ξ, component of the Hamilton vector field to recover the unknown wave speed. This in turn involves the position, x, derivative of the effective Hamiltonian. Concretely, see [20, Section 7.2.2], the ξ-component of this formula for two Hamilton vector fields H_p and $H_{\tilde{p}}$ corresponding to two effective Hamiltonians p and $\tilde{p}$, denoting their flows by (X, Ξ) (with corresponding integral curve γ, exit time $\tau = \tau(x, \xi)$), resp. $(\tilde{X}, \tilde{\Xi})$, with $f = p - \tilde{p}$, takes the form

$$\begin{aligned} J_i f(\gamma) = \int (A_i^j(X(t), \Xi(t))\partial_{x^j} f(X(t), \Xi(t)) \\ + B_{ij}(X(t), \Xi(t))\partial_{\xi_j} f(X(t), \Xi(t)))\,\mathrm{d}t = 0 \end{aligned} \tag{3.1}$$

with

$$\begin{aligned} A_i^j(x, \xi) &= -\frac{\partial \tilde{\Xi}_i}{\partial \xi_j}(\tau(x, \xi), (x, \xi)), \\ B_{ij}(x, \xi) &= \frac{\partial \tilde{\Xi}_i}{\partial x^j}(\tau(x, \xi), (x, \xi)). \end{aligned}$$

Thus, at the boundary,

$$A_i^j(x, \xi) = -\delta_i^j, \;\; B_{ij}(x, \xi) = 0.$$

Now suppose there is a function $P = P(x, \xi, \nu_1, \dots, \nu_k)$ depending on parameters ν_j, which are here the material parameters a_{ij}, and corresponding to either wave speed $G_{qP/qSV}$, and suppose that

$$p(x, \xi) = P(x, \xi, \nu_1, \dots, \nu_k), \;\; \tilde{p}(x, \xi) = P(x, \xi, \tilde{\nu}_1, \dots, \tilde{\nu}_k),$$

for two media with particular parameters $\nu_1, \dots, \nu_k$, resp. $\tilde{\nu}_1, \dots, \tilde{\nu}_k$. Then

$$p(x, \xi) - \tilde{p}(x, \xi) = \sum_{j=1}^{k} (\nu_j(x) - \tilde{\nu}_j(x)) E^j(x, \xi),$$

with

$$E^j(x, \xi) = \int_0^1 \frac{\partial P}{\partial \nu_j}(s\nu_1(x) + (1-s)\tilde{\nu}_1(x), \dots, s\nu_k(x) + (1-s)\tilde{\nu}_k(x), x, \xi)\,\mathrm{d}s.$$

Now, if these two media have the same lens relations (and thus locally if they simply have the same travel times), the Stefanov–Uhlmann identity gives with $f_l(x) = \nu_l(x) - \tilde{\nu}_l(x)$, and now $f = (f_1, \dots, f_k)$,

$$J_i f(\gamma) = \sum_{l=1}^{k} \int (\hat{A}_i^{jl}(X(t), \Xi(t))\partial_{x^j} f_l(X(t)) + \hat{B}_i^l(X(t), \Xi(t)) f_l(X(t))\,\mathrm{d}t = 0 \tag{3.2}$$

with

$$\hat{A}_i^{jl}(x,\xi) = -\frac{\partial \tilde{\Xi}_i}{\partial \xi_j}(\tau(x,\xi),(x,\xi))E^l(x,\xi),$$
$$\hat{B}_i^l(x,\xi) = -\frac{\partial \tilde{\Xi}_i}{\partial \xi_j}(\tau(x,\xi),(x,\xi))\partial_{x^j}E^l(x,\xi) + \frac{\partial \tilde{\Xi}_i}{\partial x^j}(\tau(x,\xi),(x,\xi))\partial_{\xi_j}E^l(x,\xi).$$

Thus, at the boundary, where the ν_l and $\tilde{\nu}_l$ are equal,

$$\hat{A}_i^{jl}(x,\xi) = -\delta_i^j \frac{\partial P}{\partial \nu_l}(\nu_1(x),\dots,\nu_k(x),x,\xi),$$
$$\hat{B}_i^l(x,\xi) = -\delta_i^j \partial_{x^j}\frac{\partial P}{\partial \nu_l}(\nu_1(x),\dots,\nu_k(x),x,\xi).$$

Notice that now this is a *linear* transform in the f_l.

A convex foliation, by level sets of a function x, often plays a role in this paper. The level sets of a function x on M are *concave*, or *concave from the superlevel sets*, or *convex from the sublevel sets* for a Hamiltonian p if along integral curves $\gamma(t) = (X(t), \Xi(t)))$ of H_p, $\frac{\mathrm{d}}{\mathrm{d}t}(\mathsf{x}\circ X)(t) = 0$ implies $\frac{\mathrm{d}^2}{\mathrm{d}t^2}(\mathsf{x}\circ X)(t) > 0$. This corresponds to the assumptions in Section 3.2 of [23] above Equation (3.1). Convexity from the superlevel sets is defined similarly, with the strict inequality reversed.

Instead of using the cotangent space for parameterizing the bicharacteristics, one may want to use the tangent space instead. For this one considers the Hamilton vector field map of the Hamiltonian function p: the tangent vector to a projected bicharacteristic $\gamma(t) = X(t)$ corresponding to the bicharacteristic $(X(t), \Xi(t))$ is $\dot{\gamma}(t) = H_{X(t)}(\Xi(t))$, where H_x is the push-forward of the Hamilton vector field to the base

$$H_x(\xi) = \sum_j \frac{\partial p}{\partial \xi_j}(x,\xi)\partial_{x_j},$$

where the notation indicates that for each base point x we consider it as a map

$$\xi \mapsto H_x(\xi).$$

When $p(x,\cdot)$ is a quadratic polynomial (that is p is a quadratic polynomial in ξ), thus for Riemannian geometry and the qSH transversely isotropic waves, this is a linear map, but in general it is nonlinear. In order to parameterize the bicharacteristics, this should be a map with a smooth inverse, at least locally along the bicharacteristics we wish to use. This holds if DH_x is invertible. Explicitly, this differential is the Hessian matrix with ij entry $\frac{\partial^2 p}{\partial \xi_i \partial \xi_j}$. If $p(x,\cdot)$ is a positive definite quadratic polynomial, such as in Riemannian geometry and qSH waves, then the Hessian matrix is positive definite, thus invertible. Positive definiteness of the Hessian corresponds to strict convexity of the level sets of p from the side of the sublevel sets. In general, for interesting examples of p arising from qSV waves in transversely isotropic materials, such as for the Greenhorn shale, see for example [16, Figure 2], the strict convexity may fail.

Since the general method of [23] uses curves that are almost tangential to the level sets of the convex foliation, and in many examples (supported by geodynamical considerations) the tangent space of the level sets of the convex foliation may lie close to the orthogonal plane to the isotropy axis, we start by remarking that under easy to formulate conditions the qSV (and qP) level sets are strictly convex there. (This is guaranteed if E^2 is not exceedingly negative, while in Earth materials, typically, $E^2 \geq 0$ [21].) In the following lemma, the tilded coordinates correspond to the transverse isotropy with the third coordinate corresponding to its axis, as in the introduction; note that the Hamilton vector field being invariantly defined, it makes no difference in what coordinates we consider the map H_x (we also recall the standing assumption $\max\{a_{55}, a_{66}\} < \min\{a_{11}, a_{33}\}$ here):

Lemma 3.1. *Suppose that either* $p = p_+ = G_{qP}$*, or instead* $p = p_- = G_{qSV}$ *and*

$$a_{33}(a_{11} - a_{55}) > (a_{13} + a_{55})^2.$$

Then the map $\tilde{\xi} \mapsto H_{\tilde{x}}(\tilde{\xi}) = \sum_j \frac{\partial p}{\partial \tilde{\xi}_j} \partial_{\tilde{x}_j}$ *has an invertible differential at* $\tilde{\xi}_3 = 0$*, and indeed the level sets of* p *are strictly convex* (*from the sublevel sets*) *nearby.*

Remark 3.1. This lemma also plays an important role below in studying the precise degeneracy in determining various material parameters from various waves.

Remark 3.2. Notice that if $E^2 \geqq 0$, the right-hand side is $\leqq (a_{11} - a_{55})(a_{33} - a_{55})$, so the inequality in the statement of the lemma is automatically true.

Proof. We just need to compute the Hessian matrix $\frac{1}{2} \frac{\partial^2 p_\pm}{\partial \tilde{\xi}_i \partial \tilde{\xi}_j}$ and show that it is positive definite when $\tilde{\xi}_3 = 0$. But this Hessian is diagonal, with a multiplicity 2 entry for the first 2 components. At $\tilde{\xi}_3 = 0$ the multiplicity two entry is particularly easy to evaluate as one may simply set $\tilde{\xi}_3 = 0$ prior to differentiation to obtain

$$(a_{11} + a_{55}) \pm (a_{11} - a_{55}),$$

which are positive. Thus it remains to evaluate the multiplicity one entry, namely $\frac{1}{2} \frac{\partial^2 p_\pm}{\partial \tilde{\xi}_3^2}$. Again, this simplifies as after the first differentiation we may set all terms with a $\tilde{\xi}_3^2$ factor to 0, that is, we just need to differentiate

$$(a_{33} + a_{55})\tilde{\xi}_3 \pm \frac{(a_{33} - a_{55})(a_{11} - a_{55})|\tilde{\xi}'|^2 - 2E^2|\tilde{\xi}'|^2}{(a_{11} - a_{55})|\tilde{\xi}'|^2} \tilde{\xi}_3,$$

which is

$$(a_{33} + a_{55}) \pm \frac{(a_{33} - a_{55})(a_{11} - a_{55}) - 2E^2}{a_{11} - a_{55}}.$$

As $E^2 = (a_{11} - a_{55})(a_{33} - a_{55}) - (a_{13} + a_{55})^2$, this simplifies to

$$(a_{33} + a_{55}) \pm \frac{2(a_{13} + a_{55})^2 - (a_{11} - a_{55})(a_{33} - a_{55})}{a_{11} - a_{55}},$$

which is

$$2a_{55} + \frac{2(a_{13} + a_{55})^2}{a_{11} - a_{55}},$$

thus always positive, for the $+$ sign, and is

$$2a_{33} - \frac{2(a_{13} + a_{55})^2}{a_{11} - a_{55}},$$

for the $-$ sign, which is positive if

$$a_{33}(a_{11} - a_{55}) > (a_{13} + a_{55})^2.$$

This completes the proof. □

In general, as already mentioned, we do not have strict convexity of the level sets, and as the Hessian changes signature, H_x ceases to have an invertible differential along some submanifolds. Globally, this results in H_x not being injective, giving rise to phenomena such as triplication (higher multiplicities cannot occur in the case of transverse isotropy) where a given (normalized) tangent vector is the image of multiple covectors. However, *for qP waves strict convexity* (*from sublevel sets*) *always holds* [2, p. 168], [13], and in general this phenomenon motivates the following definition:

Definition 3.1. A transversely isotropic material is non-degenerate relative to a convex foliation (concave from the superlevel sets for G_{qSV}) if for each point x and each vector v tangent to the convex foliation at the point x there is a covector ξ in the cotangent space over x such that $H_x(\xi) = v$ and the map H_x has invertible differential at ξ, with H_x arising from G_{qSV}. A transversely isotropic material is non-degenerate provided the statement above holds for all v (and not just v tangent to a particular convex foliation).

Lemma 3.1, under the assumed condition, thus shows that if the transverse isotropy orthogonal planes are close to the tangent spaces to a convex foliation, then the material is non-degenerate relative to the convex foliation.

In a non-degenerate, relative to a convex foliation, material, one may always consider, at least locally, the bicharacteristics to be parameterized by tangent vectors. This is useful both in order to localize to almost tangent to the convex foliation vectors and also to analyze the transform: stationary phase computations, discussed below, use the natural pairing between covectors at which principal symbols are evaluated and tangent vectors to the projected bicharacteristics being used. This approach also has the advantage of connecting better to the notation of [19,20,23].

Thus, from now on, *we assume that the material is non-degenerate relative to the fixed convex foliation.* We then define a transform $\tilde{L}$ from the cotangent space, which is a transform of the form

$$\tilde{L} = \sum_i \Psi_i L_i \Psi_i^{-1} \tilde{\phi}_i,$$

$\tilde{\phi}_i$ a cutoff supported in a region on a neighborhood of which H_x is smoothly invertible, and L_i is a transform, discussed below, from the tangent space, where the local identification Ψ_i is given by pull-back by the Hamilton vector field map H_x, and Ψ_i^{-1} is the pull-back by the local inverse H_x^{-1}. Concretely, we cover a neighborhood of the tangent space of the convex foliation with open sets O_i on each of which H_x^{-1} exists as a smooth map with image $\tilde{O}_i$ in the cotangent space, and take a corresponding partition of unity ϕ_i (so $\sum_i \phi_i = 1$ on a smaller neighborhood of the tangent space of the convex foliation), and let $\tilde{\phi}_i$ be defined as $H_x^* \phi_i$ on $\tilde{O}_i$ (with support in a compact subset of $\tilde{O}_i$) and 0 outside.

In order to avoid overburdening the notation, since all arguments below are local we drop the index i, and simply write L, and understand that H_x^{-1} refers to the localized inverse for $H_x\colon \tilde{O}_i \to O_i$.

Following [23], we use coordinates $(x_1, x_2, x_3) = (y, x) = z$ in which $x = x_3$ are the level sets are the convex foliation, and $x_3 = 0$ is the artificial boundary, We write tangent vectors as $\lambda\, \partial_x + \omega\, \partial_y$, and the projected bicharacteristic corresponding to such a tangent vector at z as $\gamma_{z,\lambda,\omega} = \gamma_{x,y,\lambda,\omega}$. One then considers an operator of the form LJ, where L is a slightly modified version of J^*, and where L cuts off at the artificial boundary, (see [23]):

$$(Lv)(z) = x^{-2} \int \chi(\lambda/x) v(\gamma_{z,\lambda,\omega})\, d\lambda\, d\omega,$$

cf. [20], the displayed equation below (3.1) (this differs from [23] in normalization). Here χ is a non-negative smooth compactly supported function, $\chi(0) > 0$, which is appropriately chosen as in [23], see also Lemma 3.7. The particular smooth measure $d\lambda\, d\omega$ is irrelevant; any other positive definite smooth measure will do. Note that the measure has nothing to do with the Euclidean metric g_0 (which plays a role in the transverse isotropy!) in particular, and similarly the coordinates x_j have nothing to do with Euclidean metric.

The main terms in (3.2) are the $\hat{A}_i^{jl}(x, \xi)$ terms; the others can be absorbed into these by Poincaré inequalities, at least *if the $\hat{A}_i^{jl}(x, \xi)$ terms are non-degenerate*, see [19]. To leading order at the boundary these decouple due to the δ_i^j, so one is essentially working on a microlocally weighted X-ray transform combining the differences of the unknown material parameters; more precisely one has a transform for each derivative of the combinations of the differences of these unknown material parameters. (One of course has to deal with these transforms together as done in [19] and follow-up papers.) Thus, one may consider the simplified transforms

$$\tilde{J} \tilde{f}(\gamma) = \sum_{l=1}^{k} \int (\tilde{A}^l(X(t), \Xi(t)) \tilde{f}_l(X(t))\, dt = 0 \tag{3.3}$$

with

$$\tilde{A}^l(x, \xi) = -\frac{\partial \tilde{\Xi}_j}{\partial \xi_j}(\tau(x, \xi), (x, \xi)) E^l(x, \xi),$$

$$\tilde{A}^l(x, \xi) = -\frac{\partial P}{\partial \nu_l}(\nu_1(x), \ldots, \nu_k(x), x, \xi) \text{ at } \partial M,$$

$$\tilde{f}_l = \partial_j f_l,$$

with j fixed.

The following proposition is proved completely analogously to [19, Corollary 3.1]:

Proposition 3.1. *If the operator $e^{-F/x} L \tilde{J} e^{F/x}$ is elliptic when considered as a map from a single component of $\tilde{f}_l$ (that is with the others set to 0), the ellipticity of the full operator $e^{-F/x} L J e^{F/x}$ follows as a map for the corresponding component, provided the artificial boundary is sufficiently close to ∂M.*

Roughly speaking, the hypothesis of this proposition says that *ignoring coupling* one can recover the derivatives of f_l (due to ellipticity, choosing the artificial boundary sufficiently close to ∂M), which then, as the conclusion states, allows one to recover f_l, though due to the coupling in LJ, a Poincaré inequality based argument is needed as in [19, Corollary 3.1]. Because of this proposition, in what follows we concentrate on properties of $L\tilde{J}$.

Now, with $p_\pm = G_{qP/qsV}$ (with + for qP; notice that $p_\pm$ stands for P above), and with tilded coordinates corresponding to the transversely isotropic structure, *not* the convex foliation, we have from (1.1) that

$$\begin{aligned}
\frac{\partial p_\pm}{\partial E^2} &= \mp \frac{2|\tilde{\xi}'|^2 \tilde{\xi}_3^2}{\sqrt{\big((a_{11}-a_{55})|\tilde{\xi}'|^2 + (a_{33}-a_{55})\tilde{\xi}_3^2\big)^2 - 4E^2 |\tilde{\xi}'|^2 \tilde{\xi}_3^2}},\\
\frac{\partial p_\pm}{\partial a_{11}} &= |\tilde{\xi}'|^2 \left(1 \pm \frac{(a_{11}-a_{55})|\tilde{\xi}'|^2 + (a_{33}-a_{55})\tilde{\xi}_3^2}{\sqrt{\big((a_{11}-a_{55})|\tilde{\xi}'|^2 + (a_{33}-a_{55})\tilde{\xi}_3^2\big)^2 - 4E^2 |\tilde{\xi}'|^2 \tilde{\xi}_3^2}}\right),\\
\frac{\partial p_\pm}{\partial a_{33}} &= \tilde{\xi}_3^2 \left(1 \pm \frac{(a_{11}-a_{55})|\tilde{\xi}'|^2 + (a_{33}-a_{55})\tilde{\xi}_3^2}{\sqrt{\big((a_{11}-a_{55})|\tilde{\xi}'|^2 + (a_{33}-a_{55})\tilde{\xi}_3^2\big)^2 - 4E^2 |\tilde{\xi}'|^2 \tilde{\xi}_3^2}}\right).
\end{aligned} \tag{3.4}$$

We summarize some immediate definiteness properties (keep in mind the background assumption that $\max\{a_{55}, a_{66}\} < \min\{a_{11}, a_{33}\}$) of these material derivatives in the following lemma:

Lemma 3.2. *We have*:

(1) $\frac{\partial p_+}{\partial a_{11}}$ *is a positive definite multiple of* $|\tilde{\xi}'|^2$*, thus is in particular non-negative.*
(2) $\frac{\partial p_+}{\partial a_{33}}$ *is a positive definite multiple of* $\tilde{\xi}_3^2$*, thus is in particular non-negative.*
(3) $\frac{\partial p_\pm}{\partial E^2}$ *are positive definite multiples of* $\mp|\tilde{\xi}'|^2\tilde{\xi}_3^2$*, thus is in particular non-positive/non-negative.*
(4) *If* $E^2 > 0$*,* $\frac{\partial p_-}{\partial a_{11}}$ *is negative definite multiple of* $|\tilde{\xi}'|^2$ *away from* $\tilde{\xi}_3 = 0$ *and from* $\tilde{\xi}' = 0$*, and is everywhere non-positive; the analogous statement holds if* $E^2 < 0$ *with 'positive' and 'negative' reversed.*
(5) *If* $E^2 > 0$*,* $\frac{\partial p_-}{\partial a_{33}}$ *is a negative definite multiple of* $\tilde{\xi}_3^2$ *away from* $\tilde{\xi}_3 = 0$ *and from* $\tilde{\xi}' = 0$*, and is everywhere non-positive; the analogous statement holds if* $E^2 < 0$ *with 'positive' and 'negative' reversed.*

Remark 3.3. Notice that when $E^2 = 0$, $\frac{\partial p_-}{\partial a_{11}} = 0$ and $\frac{\partial p_-}{\partial a_{33}} = 0$, that is, a_{11} and a_{33} affect only p_+. (In isotropic elasticity, $a_{11} = a_{33} = \lambda + 2\mu$ in terms of the Lamé parameters, while $a_{55} = \mu$, and $E^2 = 0$, $a_{13} = \lambda$, so this is the statement that the S waves are insensitive to λ.)

As we recall below, the derivatives in (3.4) will be evaluated at the points in the cotangent bundle on the support of the cutoff χ in L, which means that points near the image of the tangent space of the level sets of the convex foliation under the local inverses H_x^{-1}. Furthermore, for the principal symbol computation for $L\tilde{J}$ at a point (x, ζ), by stationary phase, one actually needs the base tangent vector to the bicharacteristic, $H_x(\xi) = \lambda\,\partial_x + \omega \cdot \partial_y$, be annihilated by $\sum_j \zeta_j \,\mathrm{d}x_j$, though this needs to be suitably interpreted at the artificial boundary since ζ is actually a scattering covector.

3.2. Principal Symbols

Concretely, for the standard principal symbol computation one writes the projected bicharacteristics through x, y, with tangent vector $\lambda\partial_x + \omega \cdot \partial_y$ at that point, in the form

$$\begin{aligned}\gamma(t) = \gamma_{x,y,\lambda,\omega}(t) &= (\gamma^{(1)}_{x,y,\lambda,\omega}(t), \gamma^{(2)}_{x,y,\lambda,\omega}(t))\\ &= (x + \lambda t + \alpha t^2 + O(t^3), y + \omega t + O(t^2)),\end{aligned}$$

where the O's are understood to mean the indicated prefactor times a smooth function of all variables,

$$(x_1, x_2, x_3) = ((x_1, x_2), x_3) = (y, x) = z,$$

where $\alpha = \alpha(x, y, \lambda, \omega)$, see Section 4 of [20], including expanding all the O error terms into smooth functions. Thus, to match the notation of this paper and [23], the scalar function x stands for x_3 in terms of the vector coordinates (x_1, x_2, x_3), and yet it is written as the first component of γ. (Technically, that section of [20] is in the more complicated one form setting, so there are various slight simplifications in the computations for the present purposes.) Then

$$(\gamma(t), \dot\gamma(t)) = \Big(x + \lambda t + \alpha t^2 + O(t^3), y + \omega t + O(t^2), \lambda + 2\alpha t + O(t^2), \omega + O(t)\Big),$$

and scaling $\hat\lambda = \lambda/x$, $\hat t = t/x$, as relevant below for the oscillatory integral, gives

$$\begin{aligned}(\gamma(x\hat t), \dot\gamma(x\hat t)) = \Big(&x + x^2(\hat\lambda\hat t + \alpha\hat t^2 + xO(\hat t^3)),\\ &y + x(\omega\hat t + xO(\hat t^2)), x(\hat\lambda + 2\alpha\hat t + xO(\hat t^2)), \omega + xO(\hat t)\Big).\end{aligned}$$

The weight $\hat A_i^{jl}$ is on the phase space due to the Hamiltonian dynamics used in the Stefanov–Uhlmann formula, so the tangent vector $\dot\gamma(t)$ needs to be converted to a covector via the local inverse $H^{-1}_{\gamma(t)}$ of the Hamilton vector field map.

The symbol whose left quantization is

$$A_{l,F} = e^{-F/x} L \tilde{J} e^{F/x}, \tag{3.5}$$

considered restricted to functions with only non-vanishing lth components, is, cf. [20, Equation (4.9)], combined with the weights discussed in [20, Equation (6.15)],

$$\begin{aligned} a_{l,F}(x, y, \zeta) = \int e^{-F/x} e^{F/\gamma^{(1)}_{x,y,\lambda,\omega}(t)} e^{i(\zeta_3/x^2, \zeta'/x)\cdot(\gamma^{(1)}_{x,y,\lambda,\omega}(t)-x, \gamma^{(2)}_{x,y,\lambda,\omega}(t)-y)} \\ \tilde{A}^l(\gamma(t), H^{-1}_{\gamma(t)}(\dot{\gamma}(t)))\chi(\lambda/x)\, \mathrm{d}t\, \mathrm{d}\lambda\, \mathrm{d}\omega, \end{aligned} \tag{3.6}$$

where ζ is the scattering coordinate (so covectors are $\zeta_3 \frac{\mathrm{d}x}{x^2} + \zeta' \cdot \frac{\mathrm{d}y}{x}$). After rescaling λ and t, this becomes a non-degenerate oscillatory integral with critical points at the codimension 2 submanifold

$$\hat{t} = 0, \ \zeta_3 \hat{\lambda} + \zeta' \cdot \omega = 0. \tag{3.7}$$

Note that if ζ' is large relative to ζ_3, that is, if we stay away from a conic neighborhood of $\zeta' = 0$, one can use $\hat{t}$ and $\omega^{\|}$ as the variables in which the stationary phase is performed, where ω is decomposed relative to ζ' into a parallel and a perpendicular vector; then $\hat{\lambda}, \omega^{\perp}$ parameterize the critical set. On the other hand, if ζ_3 is large relative to ζ', then one can use $\hat{t}$ and $\hat{\lambda}$ as the variables in which stationary phase is performed; then ω parameterizes the critical set. Hence, substituting the above expressions for $\gamma, \dot{\gamma}$, we conclude that up to errors that are $O(x\langle\zeta\rangle^{-1})$ relative to the a priori order, $(-1, 0)$, arising from the 0th order symbol in the oscillatory integral and the 2-dimensional space in which the stationary phase lemma is applied,

$$\begin{aligned} a_{l,F}(x, y, \zeta) \\ = \int e^{i(\zeta_3(\hat{\lambda}\hat{t}+\alpha\hat{t}^2)+\zeta'\cdot(\omega\hat{t}))} e^{-F(\hat{\lambda}\hat{t}+\alpha\hat{t}^2)} \tilde{A}^l(x, y, H^{-1}_{x,y}(x\hat{\lambda}, \omega))\chi(\hat{\lambda})\, \mathrm{d}\hat{t}\, \mathrm{d}\hat{\lambda}\, \mathrm{d}\omega. \end{aligned} \tag{3.8}$$

At this point we apply the stationary phase lemma. Up to an overall elliptic factor, this results in an integral of

$$\tilde{A}^l(x, y, H^{-1}_{x,y}(x\hat{\lambda}, \omega))\chi(\hat{\lambda})$$

over the critical set with a positive weight. In particular, if this has a fixed indefinite sign, for example is non-negative at all points of, and is actually definite (positive in the example) at one point of, the critical set, the resulting operator is elliptic. Note that we are using $\chi \geqq 0$ with $\chi(0) > 0$, so if $\tilde{A}^l$ has a fixed indefinite sign, the key question is if there is a point on the critical set with $\hat{\lambda}$ small at which $\tilde{A}^l$ has a definite sign.

Notice (3.7) states that if $\zeta_3 = 0$, the tangent vector ω is annihilated by ζ' and $\hat{\lambda}$ is arbitrary (with the localizing cutoff χ keeping it in a compact region); otherwise ω is actually arbitrary, though if ζ_3 is small relative to ζ', this requires a very large $\hat{\lambda}$, which may fall outside the support of the cutoff χ; in any case as long as the cutoff is non-trivial at zero, a neighborhood of those ω annihilated by ζ' is relevant.

Notice that at $x = 0$, *regardless of the value of* $\hat{\lambda}$, the corresponding tangent vector is just $(x\hat{\lambda})\partial_x + \omega \cdot \partial_y = \omega \cdot \partial_y$, cf. the argument of $\tilde{A}^l$.

Now, if one regards two of a_{11}, a_{33}, E^2 as known, and thus the value of l is fixed to be the remaining single unknown, then, corresponding to (3.2), at the artificial boundary the standard principal symbol of $A_{l,F} = e^{-F/x} L \tilde{J} e^{F/x}$ is, up to overall elliptic factors, simply an integral with a weight given by the remaining unknown's derivative, so for instance $\frac{\partial p_\pm}{\partial E^2}$ if E^2 is not known. Namely, the principal symbol of $A_{l,F} = e^{-F/x} L \tilde{J} e^{F/x}$ is an integral over all ω (except in the special case when $\zeta_3 = 0$, when only two values of ω enter) at covectors (z, ξ) where the covectors ξ are the images of $(0, \omega)$ under the map H_z^{-1}, which, as we recall, is the local inverse of H_z. Thus, if these partials are either positive at all points or negative at all points, or simply non-negative, resp. non-positive at all points with a strict sign at *one* point, the principal symbol is elliptic, since it is an integral of these expressions with respect to a smooth positive measure, up to an overall elliptic factor.

Now, for p_+ all the partials are non-zero as long as $\tilde{\xi}'$ and $\tilde{\xi}_3$ are non-zero, with $\frac{\partial p_+}{\partial E^2}$ negative, the others positive; for p_- the analogous claim holds for the E^2 partial, and in addition for the a_{11} and a_{33} partials provided that $E^2 > 0$.

For principal symbol computations we only need to consider the tangent plane to the artificial boundary (and nearby level sets in the interior); the question is whether the potential degeneracy of the weights at $\tilde{\xi}' = 0$ or $\tilde{\xi}_3 = 0$ provides an obstruction to a strict sign at least one point of relevance.

First consider the degeneracy of some of our weights, such as $\frac{\partial p_+}{\partial a_{11}}$, at $\tilde{\xi}' = 0$.

Lemma 3.3. *If the gradient ∇f of the transverse isotropy foliation function is not parallel to the artificial boundary, points with $\tilde{\xi}' = 0$ cannot give rise to vectors tangent to the artificial boundary under Hamiltonian map H_x.*

Note that the hypothesis on ∇f follows at least sufficiently close to the actual boundary under the conditions we discussed for the qSH waves, which we are assuming.

Proof. With an abuse of notation $\sum_j \frac{\partial p_\pm}{\partial \xi_j} \partial_{x_j}$ is $\sum_j \frac{\partial p_\pm}{\partial \tilde{\xi}_j} \partial_{\tilde{x}_j}$, both being the base component, that is pushforward to the base, of the Hamilton vector field, expressed in different coordinates. But if $\tilde{\xi}'$ vanishes, this vector is a multiple of $\partial_{\tilde{x}_3}$, so is parallel to the axis of transverse isotropic elasticity. Correspondingly, if the gradient ∇f of the transverse isotropy foliation function is not parallel to the artificial boundary, which we are assuming, then $\tilde{\xi}'$ cannot vanish at the relevant points as $\partial_{\tilde{x}_3}$ cannot be both parallel to the axis *and* tangent to the level sets of the convex foliation function. □

Thus, if we have a weight which is non-negative and only vanishes if $\tilde{\xi}' = 0$, such as $\frac{\partial p_+}{\partial a_{11}}$, arising when we are attempting to recover a_{11} from p_+ travel times, we indeed have ellipticity at the standard principal symbol level, that is, $A_{l,F}$ is elliptic in the standard sense when l corresponds to a_{11} and the wave speed used is p_+. *Together with Lemma 3.7 below, this proves that the qP travel times determine a_{11}, in the sense of Theorem 1.2.*

Unfortunately, there are points in the tangent plane at the artificial boundary with $\tilde{\xi}_3 = 0$, however, which is an issue for the determination of E^2 and a_{33}. Indeed, this is exactly the statement that there are points in the tangent plane annihilated by the differential $\mathrm{d}f$ of the foliation function, that is orthogonal to ∇f, which happens even under the conditions we discussed for the qSH waves. Since we still have a fixed though degenerate sign for our weight (so it can vanish, but is $\geqq 0$ everywhere or $\leqq 0$ everywhere), this is only an issue if the weight vanishes at every point at which the weight is evaluated in the principal symbol.

Note the vanishing phenomenon for the weight occurs for all relevant covectors even away from the artificial boundary.

Lemma 3.4. *Away from the artificial boundary, in $x > 0$, the only points ζ for which $\tilde{\xi}_3 = 0$ at all points on the critical set near the tangent space of the foliation giving the artificial boundary are those in the span of* $\mathrm{d}f$.

Proof. First, at covectors in the span of $\mathrm{d}f$, we are integrating over integral curves with tangent vectors annihilated by $\mathrm{d}f$, but at all of these $\tilde{\xi}_3 = 0$. On the other hand, for any covector ζ not in the span of $\mathrm{d}f$, the set of 'bad vectors' annihilated by both ζ and $\mathrm{d}f$ is a line, so in any open set of vectors annihilated by ζ (which thus form a 2-dimensional family), such as those in an arbitrarily small neighborhood of the tangent space to the level set of the foliation, there will be vectors which are *not* in the 'bad set', proving the lemma. □

Thus, *away from the boundary ellipticity can only fail at points in the span of* $\mathrm{d}f$, where we have already seen that it *does fail*. That the failure is quadratic follows simply from the fixed (degenerate) sign of the principal symbol.

We now turn to the non-degeneracy of the quadratic vanishing.

Lemma 3.5. *Suppose that the hypothesis of Lemma 3.1 holds. For $A_{l,F}$ corresponding to E^2 and a_{33}, the quadratic vanishing of the principal symbol at the span of* $\mathrm{d}f$ *is non-degenerate in $x > 0$.*

Proof. The lemma follows from showing that along any line transversal to the span of $\mathrm{d}f$ the quadratic vanishing is non-degenerate, that is, the second derivative is strictly positive (or strictly negative) since then the a priori positive (or negative) indefinite nature of Hessian combined with this fact implies positive (or negative) definiteness.

But this can be seen as follows: consider ν not in the span of $\mathrm{d}f$ and $\zeta = \zeta_\varepsilon = G_0(\mathrm{d}f)^{-1}\,\mathrm{d}f + \varepsilon\nu$, where one may assume that ν is G_0-orthogonal to $\mathrm{d}f$ and of unit length; the desired non-degeneracy follows if we find a vector annihilated by ζ and close to the tangent space of the convex foliation which is the image, via the Hamilton vector field map $H_{\tilde{x}}$, of a covector $\tilde{\xi} = \tilde{\xi}_\varepsilon$ that has $|\tilde{\xi}_3| \geqq C\varepsilon$, $C > 0$ (independent of ε), for then the fact that the relevant weights are non-degenerate multiples of $\tilde{\xi}_3^2$ proves the conclusion. Since the $H_{\tilde{x}}$ maps covectors with vanishing third component to $\operatorname{Ker}\mathrm{d}f$, and $H_{\tilde{x}}$ has, by Lemma 3.1, invertible differential at $\tilde{\xi}_3 = 0$, we see that for a vector v' there is a covector mapped to it by $H_{\tilde{x}}$ whose the third (tilded) component is a non-degenerate multiple of the distance (with respect to any positive definite inner product on the tangent space) of v' from the kernel of

$\mathrm{d}f$. Hence, it suffices to show that Ker ζ contains a vector v' which is $\geqq C\varepsilon, C > 0$, distance from Ker $\mathrm{d}f$ but is still near the tangent plane to the convex foliation (so that it is within the support of the cutoff).

This final claim can be seen as follows: by linear independence, for $\varepsilon \neq 0$, Ker $\mathrm{d}f$ and Ker ζ_ε intersect in a line in an angle $\sim \varepsilon$ (more precisely, the tangent of the angle is ε), and in any compact 'annulus' (closed ball minus a smaller open ball) centered at a point in Ker $\mathrm{d}f$ at a fixed distance from the intersection, the distance between a point in Ker ζ_ε and Ker $\mathrm{d}f$ is bounded below by $C\varepsilon$, $C > 0$ (and above by a similar expression), so consider a non-zero vector v in the tangent space of the convex foliation which is annihilated by $\mathrm{d}f$; the tangent space of the foliation is 2-dimensional and $\mathrm{d}f$ is not conormal to the level sets of the convex foliation, so the span of v is well-defined (that is, there is no freedom of choice as far as the span of v is concerned); in any fixed ball around it there is then a vector v_ε in Ker ζ_ε which is distance bounded below by $C\varepsilon$ (and above by a similar expression) from Ker $\mathrm{d}f$, proving the claim. $\square$

At the artificial boundary a bit more care is required, and it requires an explicit discussion of scattering covectors and maps related to them. Since elsewhere in the paper only the statement of the lemma is used, we do not recall the background here in more detail, but see for instance [19, 23]. The argument presented below is a modification, keeping track of potential degeneracies in identifications, of the arguments discussed above for the case of points away from the artificial boundary.

Lemma 3.6. *Suppose that the hypothesis of Lemma 3.1 holds. For $A_{l,F}$ corresponding to E^2 and a_{33}, the quadratic vanishing of the principal symbol is non-degenerate near the artificial boundary as well.*

Proof. Consider a scattering covector $\zeta = \zeta_3 \frac{\mathrm{d}x}{x^2} + \zeta' \cdot \frac{\mathrm{d}y}{x}$ which is not in the span of $x_3^{-1}\,\mathrm{d}f$, x_3 the convex level set function defining the boundary. If $\zeta_3 \neq 0$, the integral giving the principal symbol contains contributions corresponding to the whole tangent space of the boundary, which cannot lie completely in the kernel of $\mathrm{d}f$ since $\mathrm{d}f$ is not conormal to the artificial boundary, so there are points at which the weight is evaluated but $\tilde{\xi}_3 \neq 0$. On the other hand, if $\zeta_3 = 0$, that is, we are working with a scattering cotangent vector which is scattering cotangent to the boundary, then as already discussed, there is a line, given by the kernel of ζ', within the tangent space of the boundary within which the weights get evaluated; this line needs to be in the kernel of $\mathrm{d}f$ to lose ellipticity. But the kernel of $\mathrm{d}f$ within the tangent space to the boundary is also one dimensional (since $\mathrm{d}f$ is not conormal to the boundary), and includes the kernel of the (non-zero!) projection of $\mathrm{d}f$, so it is exactly the latter. Thus, ellipticity fails exactly if these two are the same, that is exactly if ζ is a multiple of the image of $x_3^{-1}\,\mathrm{d}f$ in the scattering cotangent bundle.

Again, the quadratic nature of the vanishing of the principal symbol follows from the fixed, though degenerate, sign of the principal symbol.

Finally, the non-degeneracy of the quadratic vanishing can be seen by an argument broadly similar to the one given above away from the boundary. Although we have scattering pseudodifferential operators to consider, so their standard principal symbols are homogeneous functions on the scattering cotangent bundle with the zero section removed, it is beneficial to work on the b-cotangent bundle: the two are related by a conformal rescaling by x_3, so the cosphere bundles are exactly the same, and utilizing the b-cotangent bundle we spare ourselves from explicitly writing x_3 or x_3^{-1} in many places: the relevant identification is the map $\pi\colon T^*M \to {}^{\mathrm{b}}T^*M$ which is the adjoint of the smooth linear bundle map $\iota\colon {}^{\mathrm{b}}TM \to TM$ which at ∂M regards a vector tangent to ∂M as simply a vector in $T_{\partial M}M$, and thus is neither injective (the kernel is the span of $x_3\partial_{x_3}$) nor surjective at the boundary. (The scattering analogues, used in the first paragraph of the proof, are $x_3^{-1}\pi$ and ιx_3^{-1}.) Now, at each point q in ∂M, $\pi(\mathrm{d}f)$ is an element of $T_q^*\partial M$ (a well-defined subspace of ${}^{\mathrm{b}}T_q^*M$, unlike the case of T_q^*M, within which the conormal bundle is well-defined!) thus annihilates ${}^{\mathrm{b}}N_qM$ (a well-defined subspace of ${}^{\mathrm{b}}T_qM$ which is spanned by $x_3\partial_{x_3}$), so the image under ι of $\operatorname{Ker}\pi(\mathrm{d}f)$ is a line in T_qM contained in $T_q\partial M$; this is the line spanned by any nonzero vector v in $\operatorname{Ker}(\mathrm{d}f) \cap T_q\partial M$. We again consider a family $\zeta_\varepsilon = \pi(\mathrm{d}f)+\varepsilon\nu$ where $\nu \in {}^{\mathrm{b}}T_q^*M$, and we may assume that ν is orthogonal to $\pi(\mathrm{d}f)$ with respect to an inner product (dual b-metric) on ${}^{\mathrm{b}}T_q^*M$. Then $\operatorname{Ker}\zeta_\varepsilon$ and $\operatorname{Ker}\pi(\mathrm{d}f)$ again meet in a line in an angle $\sim \varepsilon$ (for $\varepsilon \neq 0$, and the angle is with respect to the aforementioned b-metric, though the $\sim \varepsilon$ statement is independent of the choice of the b-metric), and the localization via the cutoff means that we are working in a fixed small neighborhood of $T_q^*\partial M$ in ${}^{\mathrm{b}}T_q^*M$, which in particular includes a neighborhood of the aforementioned v, in which, completely similarly to above, in any fixed annulus (with respect to the b-metric) there are points v_ε in $\operatorname{Ker}\zeta_\varepsilon$ which are distance $\sim \varepsilon$ from $\operatorname{Ker}\pi(\mathrm{d}f)$. Since $\operatorname{Ker}\pi(\mathrm{d}f)$ contains the kernel of ι, the image under ι of v_ε is still $\sim \varepsilon$ distance away from $\iota(\operatorname{Ker}(\pi(\mathrm{d}f)))$. But this then finally implies that $\iota(v_\varepsilon)$ is distance $\sim \varepsilon$ away from $\operatorname{Ker}\mathrm{d}f$ itself, since this is a plane intersecting $T_q\partial M$ the line $\iota(\operatorname{Ker}(\pi(\mathrm{d}f)))$ in a fixed non-zero angle. This shows that the $\tilde{\xi}_3$ component of the covector corresponding to $\iota(v_\varepsilon)$ is $\geqq C\varepsilon$, $C > 0$, which proves the non-degeneracy as in the case away from the boundary. □

We also need to have an elliptic boundary principal symbol at finite points.

Lemma 3.7. *Suppose that the gradient ∇f of the anisotropic layer function $f = \tilde{x}_3$ is neither parallel nor orthogonal to the artificial boundary. The boundary principal symbol for determining any one of a_{11}, a_{33}, E^2 from p_+, as well as for determining E^2 from p_-, is elliptic at finite points. For determining one of a_{11}, a_{33} from p_- the corresponding statement holds if $E^2 > 0$.*

Proof. For this we recall the computation from [23] in the form used in [19, Proof of Lemma 3.5]. For this, one again writes the projected bicharacteristics in the form

$$(x + \lambda t + \alpha t^2 + O(t^3),\, y + \omega t + O(t^2)),$$

where $\alpha = \alpha(x, y, \lambda, \omega)$. Further more, one computes the integral (3.8) at $x = 0$ with a Gaussian weight function in place of χ (which one eventually approximates by a compactly supported χ) with the parameter ν, which we choose to be $\nu =$

$\digamma^{-1}\alpha$, $\digamma > 0$ to be chosen sufficiently large. This gives, for example for E^2,

$$(\zeta_3^2 + \digamma^2)^{-1/2} \int_{\mathbb{S}^1} \nu^{-1/2} e^{-(\hat{Y}\cdot\zeta')^2/(2\nu(\zeta_3^2+\digamma^2))} \frac{\partial p_\pm}{\partial E^2}(x,\xi)\,\mathrm{d}\hat{Y},$$

where the covector ξ is the image of the tangent vector $(\lambda, \omega) = (0, \hat{Y})$ under H_x^{-1}, the local inverse of H_x. Unlike for the case of the standard principal symbol, for which a stationary phase computation was used, here there is no critical set to restrict to, that is, we are integrating with $\frac{\partial p_\pm}{\partial E^2}$ evaluated at the images of *all* tangent vectors to the artificial boundary. This expression is positive, resp. negative, if $\frac{\partial p_\pm}{\partial E^2}(x,\xi) \geqq 0$, resp. $\leqq 0$, for all relevant ξ, with the inequality definite for at least one of them; the relevant ξ are the images of $(0, \omega)$ under H_x^{-1}. But, taking into account (3.4), this is the case for both p_+ and p_-, with the definiteness coming from $\tilde{\xi}'$ and $\tilde{\xi}_3$ both being non-zero at one such image, since the non-parallel nature of ∇f to the artificial boundary means that $\tilde{\xi}'$ indeed never vanishes on the preimage, while the vanishing of $\tilde{\xi}_3$ at a point would mean that the corresponding tangent vector is orthogonal to ∇f, which in turn cannot happen everywhere as ∇f is not orthogonal to the artificial boundary. A completely analogous conclusion holds for $\frac{\partial p_+}{\partial a_{11}}(x,\xi)$ and $\frac{\partial p_+}{\partial a_{33}}(x,\xi)$, and if in addition $E^2 > 0$, also for $\frac{\partial p_-}{\partial a_{11}}(x,\xi)$ and $\frac{\partial p_-}{\partial a_{33}}(x,\xi)$. $\square$

The conclusion is that, with the others taken as known, the operator $e^{-\digamma/x} L \tilde{J} e^{\digamma/x}$ is elliptic at finite points for any one of E^2, a_{11}, a_{33} for the qP-travel time data, and E^2 (as well as a_{11}, a_{33} if $E^2 > 0$) from the qSV-travel time data, while the standard principal symbol ellipticity holds for a_{11} from the qP-travel time data. Hence, taking into account Proposition 3.1, a_{11} can be recovered from the qP-travel time data under the hypothesis that the anisotropic layers are not aligned with the convex foliation. This proves Theorem 1.2.

Corollary 1.2 is a simple extension of this.

Proof of Corollary 1.2. We suppose that there is a functional relationship between a_{11}, a_{33}, E^2, concretely, $a_{33} = F(a_{11})$ and $E^2 = H(a_{11})$, with F, H smooth, and suppose that $F' \geqq 0$. We claim that then the qP and qsV travel times determine a_{11} (and thus all the others).

To show this, we take the sum of the qP and qSV travel times. This cancels the $\pm$ in the equations (3.4), as we compute below. Namely, the effective coefficient in the pseudolinearization for a_{11} becomes

$$\begin{aligned}
&\frac{\partial p_+}{\partial a_{11}} + \frac{\partial p_+}{\partial a_{33}} F'(a_{11}) + \frac{\partial p_+}{\partial E^2} H'(a_{11}) \\
&\quad + \frac{\partial p_-}{\partial a_{11}} + \frac{\partial p_-}{\partial a_{33}} F'(a_{11}) + \frac{\partial p_-}{\partial E^2} H'(a_{11}) \\
&= 2|\tilde{\xi}'|^2 + 2F'(a_{11})\tilde{\xi}_3^2 \geqq 2|\tilde{\xi}'|^2,
\end{aligned}$$

so the above argument for recovering a_{11} given the other parameters works equally well. Indeed, if $F' > 0$ then the right-hand side can be replaced by $2|\tilde{\xi}|^2$, so in fact the above argument for a_{11} can be shortened somewhat. Note that there is no need for assuming anything about the derivative of H, only the existence of such a functional relationship, since H cancels from the computation of the pseudolinearization coefficient at the boundary. (Do note however that overall, H enters into the pseudolinearization formula, so the *existence* of H is crucial.) □

4. Determining More Than One Parameter at a Time

Of course, one would like to determine more than one of these ideally. Since we have two linear transforms, given by $p_\pm$, and since at some covectors the standard principal symbol behavior of these transforms only involves evaluation of the material derivative of $p_\pm$ at two points with identical behavior (antipodal), even with further modifications, as mentioned above in analogy with the tensor transform, *one cannot expect to recover all three in an elliptic manner*. However, it is reasonable to recover two of the three (or all three if two determine the third in a suitable manner); for this one needs a linear independence statement for the principal symbols which now must be considered a 2 by 2 matrix, with the inputs being the material parameters, the outputs the data for the different wave types p_+ versus p_-. (One will need slightly more to implement this, again cf. the modifications of the transform mentioned above.) For instance, $\frac{\partial p_\pm}{\partial E^2}(x,\xi)$ (with $+$ considered the first row, $-$ the second row of a column vector) and either $\frac{\partial p_\pm}{\partial a_{11}}(x,\xi)$ or $\frac{\partial p_\pm}{\partial a_{33}}(x,\xi)$ are certainly linearly independent as long as $\tilde\xi_3 \neq 0$ and $\tilde\xi' \neq 0$ since the two expressions $\frac{\partial p_\pm}{\partial E^2}(x,\xi)$ are negatives of each other, which is not the case for the other ones. In order to implement this, one defines L as

$$Lv(z) = x^{-2}\int \chi(\lambda/x)\begin{pmatrix} C_{1+}(z,\lambda,\omega) & C_{1-}(z,\lambda,\omega)\\ C_{2+}(z,\lambda,\omega) & C_{2-}(z,\lambda,\omega)\end{pmatrix}\begin{pmatrix} v_+(\gamma^+_{z,\lambda,\omega})\\ v_-(\gamma^-_{z,\lambda,\omega})\end{pmatrix}\,d\lambda\,d\omega,$$

where the first index of $C_{i\pm}$ refers to the parameter being recovered (first vs. second) and $\pm$ to the type of wave being used. Calling the parameters μ_1 and μ_2, we need that the integral in ω of

$$\begin{pmatrix} C_{1+}(z,0,\omega) & C_{1-}(z,0,\omega)\\ C_{2+}(z,0,\omega) & C_{2-}(z,0,\omega)\end{pmatrix}\begin{pmatrix} \frac{\partial p_+}{\partial\mu_1}(z,\xi) & \frac{\partial p_+}{\partial\mu_2}(z,\xi)\\ \frac{\partial p_-}{\partial\mu_1}(z,\xi) & \frac{\partial p_-}{\partial\mu_2}(z,\xi)\end{pmatrix}$$

over the circle with a positive weight is elliptic; here $\xi=\xi(z,\omega)$ is determined from $(0,\omega)$ as above. Now we can choose the C matrix to be simply the transpose of the second, material sensitivity matrix, at the actual boundary (where it is known!), and extend in a smooth manner into the interior. Then the integrand is positive definite over the boundary except where $\tilde\xi'=0$ or $\tilde\xi_3=0$ (where it vanishes), thus has positive definite symmetric part even nearby in the interior, thus the integral also has positive definite symmetric part. This proves the ellipticity of the boundary principal symbol at finite points, provided

$$\begin{pmatrix} \frac{\partial p_+}{\partial\mu_1}(z,\xi) & \frac{\partial p_+}{\partial\mu_2}(z,\xi)\\ \frac{\partial p_-}{\partial\mu_1}(z,\xi) & \frac{\partial p_-}{\partial\mu_2}(z,\xi)\end{pmatrix}$$

is full-rank, which holds, as discussed already, for example if $\mu_1 = E^2$, μ_2 one of the other parameters. We reiterate that if for example one assumes that a_{33} is a function of a_{11} and E^2 (rather than a priori known), very similar arguments work; in this case, assuming for example $a_{33} = \phi(a_{11})$, for the sake of an example, one simply needs that

$$\begin{pmatrix} \frac{\partial p_+}{\partial E^2}(z,\xi) & \frac{\partial p_+}{\partial a_{11}}(z,\xi) + \frac{\partial p_+}{\partial a_{33}}(z,\xi)\phi' \\ \frac{\partial p_-}{\partial E^2}(z,\xi) & \frac{\partial p_-}{\partial a_{11}}(z,\xi) + \frac{\partial p_-}{\partial a_{33}}(z,\xi)\phi' \end{pmatrix}$$

is full rank, which is the case if $\phi' > 0$.

In summary, the problem of determining E^2 and either one of a_{11} and a_{33} (or both if there is an a priori known relationship between them) from the qP and qSV data under the hypothesis that the anisotropic layers are not aligned with the convex foliation is always elliptic at finite points, and ellipticity fails only at scattering covectors aligned with the projection of the tilt axis to the boundary as well as at span of the tilt axis interior.

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