Spectral rigidity for spherically symmetric manifolds with boundary

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A R T I C L E   I N F O

A B S T R A C T

We prove a trace formula for three-dimensional spherically symmetric Riemannian manifolds with boundary which satisfy the Herglotz condition: Under a "clean intersection hypothesis" and assuming an injectivity hypothesis associated to the length spectrum, the wave trace is singular at the lengths of periodic broken rays. In particular, the Neumann spectrum of the Laplace–Beltrami operator uniquely determines the length spectrum. The trace formula also applies for the toroidal modes of the free oscillations in the earth. Under this hypothesis and the Herglotz condition, we then prove that the length spectrum is rigid: Deformations preserving the length spectrum and spherical symmetry are necessarily trivial in any dimension, provided the Herglotz condition and a geometrical condition are satisfied. Combining the two results shows that the Neumann spectrum of the Laplace–Beltrami operator is rigid in this class of manifolds with boundary.

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R É S U M É

Nous prouvons une formule de trace pour les variétés riemanniennes tridimension-nelles sphériquement symétriques à bord qui satisfont la condition de Herglotz : Sous une "hypothése d'intersection propre" et en supposant une hypothèse d'injectivité du spectre des longueurs, la trace d'onde est singulière aux longueurs des rayons brisés périodiques. En particulier, le spectre de Neumann de l'opérateur de Laplace–Beltrami détermine uniquement le spectre des longueurs. La formule de trace s'applique également aux modes toroïdaux des oscillations libres de la terre. Sous cette hypothèse et la condition de Herglotz, on prouve alors que le spectre des longueurs est rigide : Les déformations préservant le spectre des longueurs et la symétrie sphérique sont nécessairement triviales en toute dimension, pourvu que la condition de Herglotz et une condition géométrique soient satisfaites. La

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combinaison des deux résultats montre que le spectre de Neumann de l'opérateur de Laplace–Beltrami est rigide dans cette classe de variétés à bord.

1. Introduction

We establish spectral rigidity for spherically symmetric manifolds with boundary. We study the recovery of a (radially symmetric Riemannian) metric or wave speed rather than an obstacle. In addition, we allow certain boundaries that are not geodesically convex so one cannot readily apply the results of [30] where convexity was an important requirement that allowed them to use a wave equation parametrix. We require the so-called Herglotz condition while allowing an unsigned curvature; that is, curvature can be everywhere positive or it can change sign, and we allow for conjugate points. Spherically symmetric manifolds with boundary are models for planets, the preliminary reference Earth model (PREM) being the prime example. Specifically, restricting to toroidal modes, our spectral rigidity result determines the shear wave speed of Earth's mantle in the rigidity sense. There are several geometric assumptions we make that we shall enumerate here:

(A1) "Clean intersection hypothesis" (See Section 2.5). Cleanness of the fixed point sets of the billiard flow (bicharacteristic flow). This is equivalent with what we call the "periodic conjugacy condition" (see Definition 4.2) when the Herglotz condition (condition (A4) below) is satisfied (see Remark 4.3).

(A2) "Geometric spreading injectivity condition" (Definition 2.2). This is a condition to ensure that for each period, if there are two distinct geodesics with that period that are neither time reversals or rotations of one another, their principal contributions to the trace do not coincide.

(A3) "Countable conjugacy condition" (Definition 4.1).

(A4) Herglotz condition $\frac{d}{dr}\frac{r}{c(r)} > 0$.

These assumptions allow us to prove that the singular support of the wave trace is identical to the length spectrum. Assumption (A1) is a standard assumption when calculating the trace singularity by a stationary phase method to ensure that the critical manifolds are non-degenerate (see [23,30]). A ubiquitous issue in computing a trace formula is the possibility of cancellations between the contributions of two components of the same length that are not time reversals of each other to the wave trace. One usually assumes "simplicity" of the length spectrum so that any two rays with a given period are either rotations of each other or time reversals of each other, but since our trace formula computation is more explicit, we have a slightly weaker assumption (A2) to take care of this issue. Assumption (A4) allows us to construct the eigenfunctions to principal order globally, the length function and the epicentral distance have smoothness properties we use under (A4), and it prevents a certain degeneracy in the rays. Without (A4), the manifold would trap some geodesics inside and the geometry of broken rays would be very different and there would be no well-behaved global one-parameter description of the rays. Assumptions (A1), (A2), and (A4) are needed for the trace formula (Proposition 2.3), and all four assumptions are needed for spectral rigidity (Theorem 1.3), while only assumptions (A3) and (A4) are used to prove length spectral rigidity (Theorem 1.2). In Appendix C, we show that the class of metrics satisfying all of our assumptions is nonempty. Below, we provide a chart for easy reference regarding which assumptions are needed for each theorem:

	(A1)	(A2)	(A3)	(A4)
Trace formula	X	X		X
Length spectral rigidity			X	X
Spectral rigidity	X	X	X	X

The general problem of finding a manifold from spectral data is old; see e.g. the famous question [37] by Mark Kac in 1966. In 1972 he showed that the spectrum of a rotational solid allows one to determine lateral surface area [36]. The classic result closest to the present paper is that of Guillemin and Kazhdan [29] connecting spectral rigidity with periodic ray transforms on negatively curved manifolds. One approach to spectral problems of this nature is Weyl's law [35] and its many variants. Semiclassical variants of the law have proven useful for inverse problems since recovering the metric or a potential from a parameterized spectrum associated to a semiclassical operator depending on a parameter h is much simpler than from a single spectrum, although the potentials are allowed to be more complicated; see e.g. [22,13,32]. For a more detailed description of progress on similar problems, we refer the readers to [26,25,19].

To ensure that our model includes planets, we need to have a boundary and rotational symmetry. There are a number of results in similar settings. On surfaces of revolution the symmetry group is simply S^1, and with suitable assumptions the spectral inverse problem can be reduced to the study of 1D Sturm-Liouville operators due to Marchenko [8]. Stroock [44] studies a surface of revolution with boundary and uses the Weyl law to determine a 2D surface of revolution. Zelditch [47] recovers metrics of revolution on S^2 from the spectrum of the Laplace-Beltrami operator within a certain class. Gurarie [32] solves the inverse spectral problem for surfaces of revolution under the assumption that one knows the joint spectrum in $\mathbb{R}^2$ of Laplace eigenvalues and rotational eigenvalues. Our data comprise the spectrum of the Laplace-Beltrami operator on a spherically symmetric 3D manifold with boundary. We consider spherical symmetry in one more dimension, and something analogous to 1D Sturm-Liouville operators appears with respect to the radial coordinate.

A key novelty of the present paper is that the final proof is essentially a length spectral rigidity result. On a general Riemannian manifold the very existence of closed orbits is a complicated matter [3,4,28], and having a boundary makes the problem even harder. In some geometries, such as negatively curved closed manifolds, periodic orbits are known to exist and indeed spectral rigidity has been proven on such manifolds [17,16]. In the case of spherical symmetry we only need a very mild condition to ensure that there are enough periodic orbits and they behave cleanly enough; this is called the countably conjugacy condition. Thus, we only require assumptions (A2) and (A4) to show length spectral rigidity. We expect the condition to be generic on the class of manifolds we study, but verifying it is not a simple check, so we leave it as a conjecture:

Conjecture 1.1. *Assumptions (A1), (A2), and (A3) are generic among metrics that satisfy the Herglotz condition.*

The length spectrum is of interest in itself, as it is accessible from some measurements without spectral information and is relevant for geophysics. A pure spectral rigidity approach without a length spectrum (as in [47,32,20,9]) result may allow for considerably weaker assumptions, but we are not aware of such a proof for our result. Previously known methods need to be adjusted and extended significantly to accommodate the physically relevant situation in 3D with spherical symmetry and two boundary components, one of which is convex and the other concave.

The method of proof relies on a trace formula, relating the spectrum of the manifold with boundary to its length spectrum, and the injectivity of the periodic broken ray transform. Specifically, our manifold is the Euclidean annulus $M = \bar{B}(0,1) \setminus \bar{B}(0,R) \subset \mathbb{R}^n$, $R > 0$ and $n \geq 2$, with the metric $g(x) = c^{-2}(|x|)e(x)$, where e is the standard Euclidean metric and $c \colon (R, 1] \to (0, \infty)$ is a function satisfying suitable conditions. In appendix E we clarify that any spherically symmetric manifold is in fact of the form we consider – radially conformally Euclidean. Our assumption is that the Herglotz condition $\frac{\mathrm{d}}{\mathrm{d}r}(r/c(r)) > 0$ is satisfied everywhere. This condition was first discovered by Herglotz [33] and used by Wiechert and Zoeppritz [46].

By a maximal geodesic we mean a unit speed geodesic on the Riemannian manifold (M, g) with endpoints at the outer boundary $\partial M := \partial B(0,1)$. A broken ray or a billiard trajectory is a concatenation of maximal
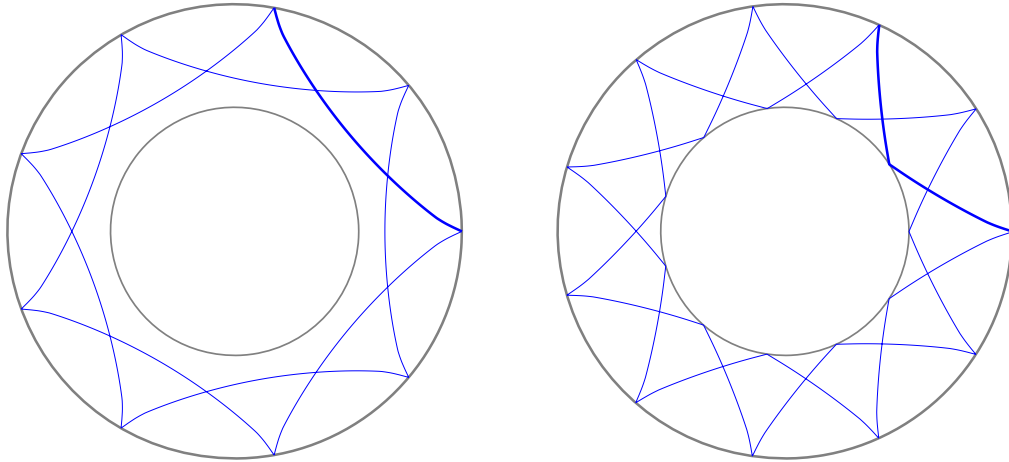
Fig. 1. The first figure shows a closed orbit whose period is in lsp(c) and lsp$'$(c). We refer to such rays as *turning rays*. The second figure shows a closed orbit whose period is in lsp$'$(c), but not in lsp(c).

geodesics satisfying the reflection condition of geometrical optics at both inner and outer boundaries of M. If the initial and final points of a broken ray coincide at the boundary, we call it a periodic broken ray – in general, we would have to require the reflection condition at the endpoints as well, but in the assumed spherical symmetry it is automatic. We will describe later (Definition 4.1) what will be called the *countable conjugacy condition* which ensures that up to rotation only countably many maximal geodesics have conjugate endpoints. For the trace formula, we will require the *clean intersection hypothesis* (see [30, Section 1]), which is that the periodic broken rays form "clean" submanifolds of loop space in the sense of Morse theory.

The length spectrum of a manifold M with boundary is the set of lengths of all periodic broken rays on M. If M is a spherically symmetric manifold as described above, we may choose whether or not we include the rays that reflect on the inner boundary $r = R$. If the radial sound speed is c, we denote the length spectrum without these interior reflections by lsp(c) and the one with these reflections by lsp$'$(c). Elements of lsp$'$(c) correspond to what we call *turning rays*, which are geodesics that never intersect the inner boundary. A smooth turning ray connects two points at the outer boundary and remains in the interior of M away from these two points. See Fig. 1 for an illustration of these two types of geodesics. If the inner radius is zero ($R = 0$), the manifold is essentially a ball and the two kinds of length spectra coincide. We note that every broken ray is contained in a unique two-dimensional plane in $\mathbb{R}^n$ due to symmetry considerations. Therefore, it will suffice to consider the case $n = 2$; the results regarding geodesics and the length spectrum carry over to higher dimensions.

We denote the Neumann spectrum of the Laplace–Beltrami operator in three dimensions, $\Delta_c = c^3 \nabla \cdot c^{-1} \nabla$, on M by spec(c), where we impose Neumann-type boundary conditions on both the inner and outer boundary. The spectrum spec(c) includes multiplicity, not just the set spectrum. The trace formula is proved in dimension $n = 3$ so theorems regarding the spectrum will be stated in this dimension. The length spectral rigidity results hold in any dimension $n \geq 2$.

Some earlier results in tensor tomography the methods of which are related to ours may be found in [40–43]. Two of the main theorems we prove on spectral rigidity are the following:

Theorem 1.2. *Let $B = \bar{B}(0,1) \setminus \bar{B}(0,R) \subset \mathbb{R}^n$, $n \geq 2$ and $R \geq 0$, be an annulus (or a ball if $R = 0$). Fix $\varepsilon > 0$ and let c_τ, $\tau \in (-\varepsilon, \varepsilon)$, be a $C^{1,1}$ function $[R,1] \to (0,\infty)$ satisfying the Herglotz condition and the countable conjugacy condition and depending C^1-smoothly on the parameter τ. If $R = 0$, we assume $c_\tau'(0) = 0$. If lsp(c_τ) = lsp(c_0) for all $\tau \in (-\varepsilon, \varepsilon)$, then $c_\tau = c_0$ for all $\tau \in (-\varepsilon, \varepsilon)$.*

The result holds true also for the length spectrum $\mathrm{lsp}'(c)$ *including reflections at the inner boundary, provided that* $R > 0$.

Theorem 1.3. *Let* $B = \bar{B}(0,1) \setminus \bar{B}(0,R) \subset \mathbb{R}^3$, $R \geq 0$, *be an annulus (or a ball if* $R = 0$). *Fix* $\varepsilon > 0$ *and let* c_τ, $\tau \in (-\varepsilon, \varepsilon)$, *be a* C^∞ *function* $[R,1] \to (0,\infty)$ *satisfying the Herglotz condition and the countable conjugacy condition and depending* C^1*-smoothly on the parameter* τ. *Assume also that the length spectrum satisfies the geometric spreading injectivity condition (Definition 2.2), and assume that the periodic broken rays satisfy the clean intersection hypothesis. If* $R = 0$, *we assume that all odd order derivatives of* c_τ *vanish at 0. If* $\mathrm{spec}(c_\tau) = \mathrm{spec}(c_0)$ *for all* $\tau \in (-\varepsilon, \varepsilon)$, *then* $c_\tau = c_0$ *for all* $\tau \in (-\varepsilon, \varepsilon)$.

Theorem 1.2 is restated as Theorems 4.6 (for $\mathrm{lsp}(c)$) and 4.7 (for $\mathrm{lsp}'(c)$). We also prove an analogous theorem for the union of various spectra of this kind (Theorem 4.8). We note that the dimension is irrelevant for the length spectral rigidity results; if the sound speed is fixed, the length spectrum is independent of dimension. For Theorem 1.3, we assume (A1)-(A4), while for Theorem 1.2, we only need to assume (A3) and (A4).

Using Proposition E.1, we find the following corollaries:

Corollary 1.4. *Let* $B = \bar{B}(0,1) \setminus \bar{B}(0,R) \subset \mathbb{R}^n$, $n \geq 2$ *and* $R \geq 0$, *be an annulus (or a ball if* $R = 0$). *Fix* $\varepsilon > 0$ *and let* g_τ, $\tau \in (-\varepsilon, \varepsilon)$, *be a* $C^{1,1}$*-regular non-trapping* $SO(n)$*-invariant Riemannian metric making the boundary strictly convex and satisfying the countable conjugacy condition and depending* C^1*-smoothly on the parameter* τ. *If* $\mathrm{lsp}(g_\tau) = \mathrm{lsp}(g_0)$ *for all* $\tau \in (-\varepsilon, \varepsilon)$, *then there are rotation equivariant diffeomorphisms* $\psi_\tau \colon B \to B$ *so that* $\psi_\tau^* g_\tau = g_0$ *for all* $\tau \in (-\varepsilon, \varepsilon)$.

In dimension $n \geq 3$ all rotation equivariant diffeomorphisms (diffeomorphisms commuting with the action of $SO(n)$) are radial. In dimension $n = 2$ the diffeomorphisms are also radial if the metrics are assumed to be $O(2)$-invariant.

Corollary 1.5. *Let* $B = \bar{B}(0,1) \setminus \bar{B}(0,R) \subset \mathbb{R}^3$, *and* $R \geq 0$, *be an annulus (or a ball if* $R = 0$). *Fix* $\varepsilon > 0$ *and let* g_τ, $\tau \in (-\varepsilon, \varepsilon)$, *be a* C^∞*-regular non-trapping rotation invariant Riemannian metric making the boundary strictly convex and satisfying the countable conjugacy condition and depending* C^1*-smoothly on the parameter* τ. *Assume also that the length spectrum satisfies the geometric spreading injectivity condition. If* $\mathrm{spec}(g_\tau) = \mathrm{spec}(g_0)$ *for all* $\tau \in (-\varepsilon, \varepsilon)$, *then there are radial diffeomorphisms* $\psi_\tau \colon B \to B$ *so that* $\psi_\tau^* g_\tau = g_0$ *for all* $\tau \in (-\varepsilon, \varepsilon)$.

Remark 1.6. While our main results are stated for the Laplace–Beltrami operator, they are equally valid for the spectra associated to the toroidal modes (see [18, chapter 8.6] for a precise definition) of an elastic operator. In section 2, we point out that the length spectrum may be recovered from either the spectrum of the Laplace–Beltrami operator or the toroidal mode eigenfrequencies so the results above hold when one has the spectrum associated to toroidal modes. In fact, our trace formula (Proposition 2.3) to recover the length spectrum from the spectrum is done for the more general toroidal modes and frequencies. Nevertheless, we show how our proof and result holds for the Laplace–Beltrami operator as well. We note that the spectrum of the Laplace–Beltrami operator depends on dimension.

The proofs of these theorems require three key ingredients which we elaborate in the next subsection:

- A trace formula relating the Neumann spectrum of the Laplacian to the length spectrum. This will be an analog to the trace formula appearing in [30, theorem 1] where there was no symmetry. Thus, we will only need to prove length spectral rigidity since spectral rigidity (Theorem 1.3) follows from the trace formula. Our proof also requires a careful study of the asymptotics of the eigenfunctions since we allow

a non-convex boundary when studying the annulus and so one cannot readily invoke the parametrix construction of [30].
- Sufficiently many periodic broken rays are stable under geometry-preserving perturbations of the metric and the derivative of the length of such broken rays is the periodic broken ray transform of the variation of the metric.
- The periodic broken ray transform uniquely determines a radial function.

Outline of the proof

The breakup of the paper is as follows. In section 2, we discuss the relevant partial differential operators and their eigenfunctions. We also discuss geodesics in spherical symmetry and state the trace formula that we will prove (see Proposition 2.3). Section 3 will be devoted to a proof of the trace formula. It will be necessary to use several transforms and the Debye expansion (see Appendix A.2.2) to convert the Green's function for the wave propagator written in terms of eigenfunctions to the dynamical Fourier integral operator (FIO) representation analogous to the one in [23] and [30]. Along the way, the connection from eigenfunction to geodesics becomes rather explicit. To reinforce this point, in section 3.1 we revisit the wave propagator constructed in [30] and show how all our explicit calculations relate to the abstract, geometric construction in that paper. Finally, a standard application of the method of steepest descent and stationary phase provides the leading order asymptotics for the trace.

After proving the trace formula, we prove the rigidity of the length spectrum in section 4. Together with the trace formula this implies the rigidity of the Neumann spectrum of the Laplace–Beltrami operator. We have a family of radially symmetric wave speeds c_τ parameterized by $\tau \in (-\varepsilon, \varepsilon)$. For any periodic broken ray that is locally stable under the family of deformations, the derivative of its length is the integral of the metric variation over the periodic broken ray. In the case of closed manifolds and periodic geodesics this is well known, and in the case of non-periodic broken rays this was observed in [34]. Since the length spectrum is independent of the parameter τ, these derivatives vanish. The countable conjugacy condition (Definition 4.1) guarantees that sufficiently many periodic broken rays are stable, so that we may conclude that the periodic broken ray transform of the function $\frac{d}{d\tau} c_\tau^{-2}\big|_{\tau=0}$ vanishes. It then follows from recent results of periodic broken ray tomography on spherically symmetric manifolds [21] that the function in question has to vanish. Since the function is radial, this can be seen as an injectivity result for an Abel-type integral transform. Consequently c_τ is independent of τ and the rigidity of the length spectrum and thus the spectrum follows.

Acknowledgments

2. Geodesics and eigenfunctions

In this section, we describe the relevant partial differential operators, associated eigenfunctions, and the connection between toroidal modes and the spectrum of the Laplace–Beltrami operator. We state our trace formula (Proposition 2.3) along with an important remark related to the Laplace–Beltrami operator

described in the introduction. First, let us provide a preliminary discussion of geodesics in spherically symmetric manifolds.

2.1. Geodesics in a spherically symmetric model

For the moment, we suppose that $n = 2$ and equip the annulus $M = \bar{B}(0,1) \setminus \bar{B}(0,R)$ with spherical coordinates. For a maximal geodesic we define its radius as its Euclidean distance to the origin. We let $\gamma(r)$ be the maximal geodesic of radius r which has its tip (the closest point to the origin) at the angular position $\theta = 0$.

For $r_0 \in (R, 1)$, the geodesic $\gamma(r_0)$ can be parametrized as

$$[-L(r_0), L(r_0)] \ni t \mapsto (r(t), \theta(t))$$

so that $r(0) = r_0$ and $\theta(0) = 0$. Here $L(r_0) > 0$ is the half length of the geodesic. Using the conserved quantities $c(r(t))^{-2}[r'(t)^2 + r(t)^2\theta'(t)^2] = 1$ (squared speed) and $p = p_\gamma := c(r(t))^{-2}r(t)^2\theta'(t) = r_0/c(r_0)$ (angular momentum) one can find the functions $r(t)$ and $\theta(t)$ explicitly.

Using these conserved quantities it is straightforward to show that

$$L(r) = \int\limits_r^1 \frac{1}{c(r')} \left(1 - \left(\frac{rc(r')}{r'c(r)}\right)^2\right)^{-1/2} \mathrm{d}r' = \int\limits_r^1 \frac{1}{c(r')^2\beta(r';p)} \mathrm{d}r', \tag{2.1}$$

where $\beta(r;p) := \sqrt{c^{-2}(r) - r^{-2}p^2}$. We introduce the quantity β because it will appear naturally in the asymptotic approximations of eigenfunctions in section 3 and its relation to geodesics will now be clear.

We denote $\alpha(r) := \theta(L(r))$, where θ is the angular coordinate (taking values in $\mathbb{R}$) of the geodesic $\gamma(r)$. That is, $2\alpha(r)$ is the angular distance of the endpoints of $\gamma(r)$. It may happen that $\alpha(r) > \pi$ if the geodesic winds around the origin several times. Using the invariants given above, one can also find an explicit formula for $\alpha(r)$:

$$\alpha(r) = \int\limits_r^1 \frac{rc(r')}{c(r)(r')^2} \left(1 - \left(\frac{rc(r')}{r'c(r)}\right)^2\right)^{-1/2} \mathrm{d}r' = \int\limits_r^1 \frac{p}{(r')^2\beta(r';p)} \mathrm{d}r'. \tag{2.2}$$

We will use the following lemma without mention whenever we need regularity of these functions.

Lemma 2.1. *When c is $C^{1,1}$ and satisfies the Herglotz condition, then the functions α and L are C^1 on $(R, 1)$.*

Proof. This follows from equations (35), (67), and (68) and proposition 15 in [21]. □

2.2. Geometric spreading injectivity condition

To recover the length spectrum using the trace formula, we do not actually need simplicity of the length spectrum but rather a weaker condition. We prove later (see Equation (3.5)) that the principal contribution of a ray γ with length T, has m reflections from the outer boundary, and ray parameter p, has the form

$$c\,\mathrm{i}^{N(p)}(t - T + \mathrm{i}\,0)^{-5/2}L(p)|2mp^{-2}\partial_p\alpha(p)|^{-1/2}$$

where

$$L(p) = \int\limits_{R(p)}^{1} \frac{1}{c(r')} \left(1 - \left(\frac{rc(r')}{r'c(r)}\right)^2\right)^{-1/2} \mathrm{d}r' = \int\limits_{R(p)}^{1} \frac{1}{c(r')^2 \beta(r'; p)} \mathrm{d}r',$$

where $R(p)$ is the radius of the geodesic, we have $2mL(p) = T$, and c is independent of γ. Thus, suppose we have two rays γ_1 and γ_2 with the same length and different ray parameter such that their principal contributions in the trace formula cancel each other. Let p_1, p_2 denote their respective ray parameter. Since we have $m_1 L(p_1) = m_2 L(p_2)$, this implies that

$$p_1^{-2} \partial_p \alpha(p_1) = p_2^{-2} \partial_p \alpha(p_2).$$

These terms are related to "geometric spreading" in seismic ray theory. This motivates us to adopt a hypothesis that is considerably weaker than the simplicity of the length spectrum.

Definition 2.2. We say that the length spectrum satisfies that *geometric spreading injectivity condition* if given any two periodic rays with ray parameters p_1 and p_2, where at least one of them does not have reflections at the inner boundary, that have the same period, are not rotations of each other, and are not time reversals of each other, then

$$p_1^{-2} \partial_p \alpha(p_1) \neq p_2^{-2} \partial_p \alpha(p_2).$$

Hence, under the geometric spreading injectivity condition, there will be no cancellations in the trace formula for turning rays, and we can recover the length spectrum ($\mathrm{lsp}(c)$) from the spectrum.

2.3. Toroidal modes, eigenfrequencies, and traces

We now use spherical coordinates (r, θ, ψ). Toroidal modes are precisely the eigenfunctions of the isotropic elastic operator that are sensitive to only the shear wave speed. We forgo writing down the full elastic equation, and merely write down these special eigenfunctions connected to the shear wave speed (full details with the elastic operator may be found in [18, chapter 8.6]). Analytically, these eigenfunctions admit a separation in radial functions and real-valued spherical harmonics, that is,

$$u = {}_n\mathbf{D}_l Y_l^m,$$

where

$$\mathbf{D} = U(r) \, (-k^{-1})[-\widehat{\theta}(\sin\theta)^{-1}\partial_\psi + \widehat{\psi}\partial_\theta],$$

in which $k = \sqrt{l(l+1)}$ (instead of the asymptotic Jeans relation, $k = l + \frac{1}{2}$) and U represents a radial function ($_nU_l$). In the further analysis, we ignore the curl (which signifies a polarization); that is, we think of $_n\mathbf{D}_l$ as the multiplication with $_nU_l(-k^{-1})$. In the above, Y_l^m are spherical harmonics, defined by

$$Y_l^m(\theta, \psi) = \begin{cases} \sqrt{2} X_l^{|m|}(\theta) \cos(m\psi) & \text{if } -l \leq m < 0, \\ X_l^0(\theta) & \text{if } m = 0, \\ \sqrt{2} X_l^m(\theta) \sin(m\psi) & \text{if } 0 < m \leq l, \end{cases}$$

where

$$X_l^m(\theta) = (-)^m \sqrt{\frac{2l+1}{4\pi}} \sqrt{\frac{(l-m)!}{(l+m)!}} P_l^m(\cos\theta),$$

in which

$$P_l^m(\cos(\theta)) = (-)^m \frac{1}{2^l l!} (\sin \theta)^m \left(\frac{1}{\sin \theta} \frac{d}{d\theta} \right)^{l+m} (\sin \theta)^{2l}.$$

The function, U (a component of displacement), satisfies the equation

$$[-r^{-2}\partial_r \, r^2 \mu \partial_r + r^{-2}\partial_r \, \mu r - r^{-1}\mu\partial_r + r^{-2}(-1+k^2)\mu] \, U - \omega^2 \rho U = 0, \tag{2.3}$$

where $\mu = \mu(r)$ is a Lamé parameter and $\rho = \rho(r)$ is the density, both of which are smooth. Also, $\omega = {}_n\omega_l$ denotes the associated eigenvalue. Here, l is referred to as the angular order and m as the azimuthal order.

The traction is given by

$$T(U) = \mathcal{N}U, \qquad \mathcal{N} = \mu\partial_r - r^{-1}\mu \tag{2.4}$$

which vanishes at the boundaries (Neumann condition). The radial equations do not depend on m and, hence, every eigenfrequency is degenerate with an associated $(2l+1)$-dimensional eigenspace spanned by

$$\{Y_l^{-l}, \ldots, Y_l^l\}.$$

We use spherical coordinates (r_0, θ_0, ψ_0) for the location, x_0, of a source, and introduce the shorthand notation $({}_n\mathbf{D}_l)_0$ for the operator expressed in coordinates (r_0, θ_0, ψ_0). We now write the (toroidal contributions to the) fundamental solution as a normal mode summation

$$G(x, x_0, t) = \mathrm{Re} \sum_{l=0}^{\infty} \sum_{n=0}^{\infty} {}_n\mathbf{D}_l({}_n\mathbf{D}_l)_0 \sum_{m=-l}^{l} Y_l^m(\theta, \psi) Y_l^m(\theta_0, \psi_0) \frac{e^{\mathrm{i}\,{}_n\omega_l t}}{\mathrm{i}\,{}_n\omega_l}. \tag{2.5}$$

On the diagonal, $(r, \theta, \psi) = (r_0, \theta_0, \psi_0)$ and, hence, $\Theta = 0$. Here Θ is the angular epicentral distance, cf. (3.1). We observe the following reductions in the evaluation of the trace of (2.5):

- The functions $U(r)$ are normalized, so that

$$\int_R^1 U(r)^2 \rho(r) r^2 \, dr = 1. \tag{2.6}$$

Meanwhile, the spherical harmonic terms satisfy

$$\sum_{m=-l}^{l} \iint Y_l^m(\theta, \psi)^2 \sin \theta \, d\theta \, d\psi = 2l + 1$$

(counting the degeneracies of eigenfrequencies).

- If we were to include the curl in our analysis (generating vector spherical harmonics), taking the trace of the matrix on the diagonal yields

$$\sum_{m=-l}^{l} \iint (-k^{-2}) \left| [-\widehat{\theta}(\sin \theta)^{-1} \partial_\psi + \widehat{\psi}\partial_\theta] Y_l^m(\theta, \psi) \right|^2 \sin \theta \, d\theta \, d\psi = 2l + 1.$$

From the reductions above, we obtain

$$\int\limits_M G(x,x,t)\,\rho(x)\,\mathrm{d}x = \sum_{l=0}^{\infty}\sum_{n=0}^{\infty}(2l+1)\,\mathrm{Re}\left\{\frac{e^{\mathrm{i}\,_n\omega_l t}}{\mathrm{i}\,_n\omega_l}\right\}$$

or

$$\mathrm{Tr}(\partial_t G)(t) = \int\limits_M \partial_t G(x,x,t)\,\rho(x)\,\mathrm{d}x = \sum_{l=0}^{\infty}\sum_{n=0}^{\infty}(2l+1)\,\mathrm{Re}\left\{e^{\mathrm{i}\,_n\omega_l t}\right\}. \tag{2.7}$$

We now write

$$_n f_l(t) = \mathrm{Re}\left\{\frac{e^{\mathrm{i}\,_n\omega_l t}}{\mathrm{i}\,_n\omega_l}\right\}$$

which is the inverse Fourier transform of

$$_n\hat{f}_l(\omega) = \frac{1}{2\,\mathrm{i}\,_n\omega_l}\left[\pi\delta(\omega - {}_n\omega_l) - \pi\delta(\omega + {}_n\omega_l)\right].$$

Moreover, taking the Laplace–Fourier transform yields

$$\int\limits_0^{\infty} {}_n f_l(t)e^{-\mathrm{i}\,\omega t}\,\mathrm{d}t = \frac{1}{2\,\mathrm{i}\,_n\omega_l}\left[\frac{\mathrm{i}}{-(\omega - {}_n\omega_l) + \mathrm{i}\,0} - \frac{\mathrm{i}}{-(\omega + {}_n\omega_l) + \mathrm{i}\,0}\right]. \tag{2.8}$$

This confirms that the trace is equal to the inverse Fourier transform of

$$\sum_{l=0}^{\infty}(2l+1)\sum_{n=0}^{\infty}\frac{1}{2\,\mathrm{i}\,_n\omega_l}\left[\pi\delta(\omega - {}_n\omega_l) - \pi\delta(\omega + {}_n\omega_l)\right].$$

In the next subsection, we explain how the toroidal eigenfrequencies $\{_n\omega_l\}_{n,l}$ relate to the Neumann spectrum of the Laplace–Beltrami operator described in the introduction. We also show why all our results and proofs in connection to the trace formula (Proposition 2.3) hold for this spectrum as well.

2.4. Connection between toroidal eigenfrequencies, spectrum of the Laplace–Beltrami operator, and the Schrödinger equation

We relate the spectrum of a scalar Laplacian, the eigenvalues associated to the vector valued toroidal modes, and the trace distribution $\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}(2l+1)\cos(t_n\omega_l)$.

We note that (2.3) and (2.4) for U ensure that $v = UY_l^m$ satisfies

$$Pv := \rho^{-1}(-\nabla\cdot\mu\nabla + P_0)v = \omega^2 v, \qquad \mathcal{N}v = 0 \text{ on } \partial M \tag{2.9}$$

where $P_0 = r^{-1}(\partial_r\mu)$ is a 0th order operator, ω^2 is a particular eigenvalue, and $\mathcal{N}$ is as in (2.4). Hence UY_l^m are scalar eigenfunctions for the self-adjoint (with respect to the measure $\rho\,\mathrm{d}x$) scalar operator P with Neumann boundary conditions (on both boundaries) expressed in terms of $\mathcal{N}$.

The above argument shows that we may view the toroidal spectrum $\{_n\omega_l^2\}_{n,l}$ as also the collection of eigenvalues λ for the boundary problem on scalar functions (2.9). Thus (2.7) can be written in the form

$$\mathrm{Tr}\,(\partial_t G) = \sum_{\lambda\in\mathrm{spec}(P)}\cos(t\sqrt{\lambda}),$$

where the last sum is taken with multiplicities for the eigenvalues. (While G is a vector valued distribution, the asymptotic trace formula we obtain is for $\mathrm{Tr}(\partial_t G)$, which is equal to $\sum_{\lambda \in \mathrm{spec}(P)} \cos(t\sqrt{\lambda})$ by the normalizations we have chosen.) Up to principal symbols, P coincides with the $\Delta_c = c^3 \nabla \cdot c^{-1} \nabla$ upon identifying c^2 with $\rho^{-1}\mu$. This means that the length spectra of P and Δ_c will be the same even though they have differing subprincipal symbols and spectra. Thus, the trace formula which will appear to have a unified form, connects two different spectra to a common length spectrum and the proof is identical for both.

We will prove a trace formula using a WKB expansion of eigenfunctions. To this end, it is convenient to establish a connection with the Schrödinger equation. Indeed, we present an asymptotic transformation finding this connection. In boundary normal coordinates (r, θ) (which are spherical coordinates in dimension three by treating θ as coordinates on the 2-sphere),

$$P = \rho^{-1}(-r^{-2}\partial_r r^2 \mu \partial_r - \mu r^{-2}\Delta_\theta + P_0),$$

where Δ_θ is the Laplacian on the 2-sphere. Let us now simplify the PDE (2.9) for v. Let $Y(\theta)$ be an eigenfunction of Δ_θ with eigenvalue $-k^2$ as before and $V = V(r) := \mu^{1/2} r U$ a radial function with U as in (2.9). Then after a straightforward calculation, as a leading order term in a WKB expansion, $V(r)$ must satisfy

$$\partial_r^2 V + \omega^2 \beta^2 V = 0, \quad \partial_r V = 0 \ \text{ on } \ \partial M, \tag{2.10}$$

where $\beta^2 = \rho(r)\mu(r)^{-1} - \omega^{-2}r^{-2}k^2$, generating two linearly independent solutions. The WKB asymptotic solution to this PDE with Neumann boundary conditions will precisely give us the leading order asymptotics for the trace formula, and is all that is needed.

For the boundary condition, we note that we would end up with the same partial differential equation with different boundary conditions for V in the previous section if we had used the boundary condition $\partial_r u = 0$ on ∂M. Indeed, one would merely choose $\mathcal{N}u = \mu \partial_r u$ instead without the 0th order term. However, the boundary condition for V would be of the form

$$\partial_r V = K(r)V \qquad \text{on } \ \partial M$$

with K signifying a smooth radial function. Nevertheless, the leading order (in ω) asymptotic behavior for V stays the same despite the K term as clearly seen in the calculation of section A.1.1. Thus, our analysis applies with no change using the standard Neumann boundary conditions. This should come as no surprise since in [30], the 0th order term in the Neumann condition played no role in the leading asymptotic analysis of their trace formula. Only if one desires the lower-order terms in the trace formula would it play a role.

In addition, we could also consider a Dirichlet boundary condition, where for V, it is also $V = 0$ on ∂M. This would slightly modify the quantization conditions (A.4), (A.6), (A.5), and thus affect the Debye expansion A.2.2 by constant factors. Nevertheless, the same argument holds to obtain the trace formula and recover the length spectrum. More general boundary conditions such as Robin boundary conditions may be considered as well. However, since we only need to look at the principal term in the high frequency asymptotics, this would just reduce to the Neumann boundary case. Thus, our arguments work with all these boundary conditions, and we choose Neumann boundary conditions only because it has a natural interpretation from geophysics.

2.5. Poincaré maps and the trace formula

Here, we describe the relevant Poincaré map that will appear in the trace formula and state the trace formula that we will prove. Let $\mathbb{R} \ni t \to \gamma(t)$ be a periodic broken bicharacteristic in S^*M of period $T > 0$

(see [30] for the relevant definitions). It is associated to the metric $c^{-2}(|x|)e$, where c is a smooth radial function, e is the Euclidean metric, and γ undergoes reflections in ∂S^*M according to Snell's law. We also denote by $\Phi^T\colon S^*M \to S^*M$ the broken bicharacteristic flow of T units of time as described in [30]. Its fixed point set is given by

$$C_T := \{\eta \in S^*M; \Phi^T(\eta) = \eta\}$$

and without loss of generality, we assume that C_T is connected, for otherwise, we would look at a connected component instead. We impose the *clean intersection hypothesis* appearing in [23,30] so that C_T is a submanifold for any T and $T_m C_T = \ker(\mathrm{Id} - \mathrm{d}\Phi^T(m))$ at each point $m \in C_T$. This holds for all periodic orbits if and only if c satisfies the periodic conjugacy condition (Definition 4.2) which requires that the endpoints of a single maximal geodesic segment of a periodic broken ray are never conjugate; see Remark 4.3.

By construction, the image of γ belongs to C_T. There is an obvious symplectic group action of $SO(3)$ on T^*M under which the Hamiltonian $\frac{1}{2}c^2 |\xi|^2$ is invariant; here, ξ denotes the dual variable to x. Thus, for each $g \in SO(3)$, the set $g \cdot \mathrm{Im}(\gamma)$ given by the group action also belongs to C_T. Assuming that c has no other symmetries (follows from the Herglotz condition) and the periodic conjugacy condition, all elements of C_T are obtained this way. This is because a periodic orbit will either fail to be periodic or not have period T after a small perturbation of the angular momentum p; hence, p remains constant on C_T. Thus, C_T may be parameterized by $\mathbb{R}_t \times SO(3)$, revealing that under the Herglotz and periodic conjugacy conditions

$$\dim(C_T) = 4.$$

The Herglotz condition ensures that the group action never coincides with the geodesic flow. In other words, if we consider the group action just described on S^*M (call it G) and fix $q \in S^*M$, then there is no one parameter subgroup $H \subset G$ such that the H orbit of q and the geodesic flow orbit of q coincide. Without the Herglotz condition, the dimension could quite possibly be smaller. Hence, $T_m S^*M / T_m C_T$ is only one-dimensional and we obtain an induced map on the quotient space,

$$I - (\mathrm{d}\Phi^T)^\sharp \colon T_m S^*M / T_m C_T \to \mathrm{Im}(I - (\mathrm{d}\Phi^T)^\sharp) \subset T_m C_T.$$

We denote the equivalence class of all closed orbits of period T related to γ by an element of $SO(3)$ or by a time reversal of γ by $[\gamma]$. We write the above map as $I - P_{[\gamma]}$ and refer to $P_{[\gamma]}$ as the *Poincaré map* associated to the equivalence class of γ. The clean intersection hypothesis will ensure that $I - P_{[\gamma]}$ is an isomorphism, and canonical bases may be chosen so that its determinant is a geometric quantity. This determinant at each point $m \in C_T$ will stay invariant. Hence, the quantity $\left|I - P_{[\gamma]}\right|^{-1} := \left|\det(I - P_{[\gamma]})\right|^{-1}$ is well defined as a single number associated to $[\gamma]$. This is a special quantity that only appears when one has symmetries (cf. [15], [7, Appendix]) and it corresponds to $\mathrm{d}\Phi^T$ having additional generalized eigenvectors of eigenvalue 1 (see appendix E).

In the above, γ may be multiple revolutions of another closed orbit of minimal period called the *primitive orbit* associated to γ, which has a primitive period denoted $T^\sharp$. Note that T will merely be a positive integer multiple of $T^\sharp$. In spherical symmetry, γ is confined to a disk and it must be a concatenation of geodesic segments that travel from the outer boundary $r = 1$ to either the reflection point $r = R$ or the turning point $r = R^\star$ (see section 4 for details). We let N_γ denote the number of these segments comprising the primitive orbit associate to γ.

We now state our proposition pertaining to the trace formula.

Proposition 2.3. *Suppose the radial wave speed c satisfies the Herglotz condition, the clean intersection hypothesis, and the geometric spreading injectivity condition. The distribution $(\mathrm{Tr}\,(\partial_t G))(t) = \sum_{n,l}(2l + 1)\cos(t_n \omega_l)$ is singular at $\mathrm{lsp}(c)$. Suppose $T \in \mathrm{singsupp}(\mathrm{Tr}(\partial_t G))$ and let d be the dimension of the fixed*

point set for Φ^T. *Suppose that* γ *is one of the broken periodic orbits of period* T. *Then* $d = 4$ *and for* t *near* T, *the contribution of* $[\gamma]$ *to the leading singularity of* $(\mathrm{Tr}(\partial_t G))(t)$ *is the real part of*

$$(t - T + \mathrm{i}\,0)^{-(d+1)/2} \left(\frac{1}{2\pi\,\mathrm{i}}\right)^{(d-1)/2} \mathrm{i}^{\sigma_\gamma} \left|I - P_{[\gamma]}\right|^{-1/2} \frac{T_\gamma^\sharp}{2\pi N_\gamma} c_d \left|SO(3)\right|,$$

where

- σ_γ *is the Keller-Maslov-Arnold-Hörmander (KMAH) index associated to* γ *defined in* [30];
- c_d *is a constant depending only on* d;
- $|SO(3)|$ *is the volume of the compact Lie group* $SO(3)$ *under the Haar measure.*

In appendix D we identify this trace formula in the framework of manifolds with symmetries given by a compact Lie group. In appendix F we discuss some edge cases that justify our geometric assumptions.

Remark 2.4. This trace formula is in fact a more general statement than that for the Neumann Laplace–Beltrami operator, which is just a special case of the above proposition.

Remark 2.5. We note that our trace formula holds in an annulus where the boundary is not geodesically convex unlike the case in [30]. Hence, there could be periodic *grazing rays* at the inner boundary of the annulus. As described in [45], grazing rays are bicharacteristics that intersect the boundary tangentially, have exactly second order contact with the boundary, and remain in $\bar{M}$. This is another reason our proof is via a careful study of the asymptotics of the eigenfunctions rather than the parametrix construction appearing in [30], where the presence of a periodic grazing ray would make the analysis significantly more technical (cf. [45,2]). The spherical symmetry essentially allows us to construct a global parameterix (to leading order) to obtain the leading order contribution of a periodic grazing ray to the trace, which would be more challenging in a general setting (see Appendix B for the analysis and [6] for a similar computation). The leading order contribution of the grazing ray has the same form as in the above proposition, but the lower order contributions will not have this "classical" form since stationary phase cannot be applied to such terms.

Nevertheless, we note that for the main theorems, we do not need to recover the travel time of a periodic grazing ray if one exists. Travel times of sufficiently many reflecting and turning rays suffice. Our methods also produce a precise trace formula where periodic orbits are no longer simple as in [30], but come in higher dimensional families (see [31,14,15,27] for related formulas albeit in different settings).

Remark 2.6. The clean intersection hypothesis and the geometric spreading injectivity condition are needed to prove that the singular support of the trace includes lengths of periodic broken rays. However, they are not necessary for proving length spectral rigidity.

For unique determination of the length spectrum, it is enough that the primitive length spectrum (excluding all but primitive orbits) is simple or the geometric spreading injectivity condition is satisfied. Given the singularity at $T^\sharp$, we know what the singularities at $2T^\sharp, 3T^\sharp, \ldots$ will be. If they are not as expected, then another broken ray must contribute a singularity at the same place, and we have found the next primitive length. This allows to recover the primitive length spectrum and therefore the whole length spectrum from the (shapes and locations of) singularities in the trace. But if two primitive lengths coincide, there is no way to distinguish the corresponding singularities.

If we drop the periodic conjugacy condition, then some periodic orbits may fail the clean intersection hypothesis. This allows us to recover only a part of the length spectrum from the singularities, but this part is enough. Such problematic periodic broken rays are ignored anyway in the proof of length spectral rigidity, since they might not be stable under deformations.

3. Proof of the trace formula (Proposition 2.3)

In this section, we prove the trace formula in the form of Proposition 2.3 for the annulus. The idea behind the proof is to construct rather explicitly a fundamental solution. First, we do some preliminary analysis to manipulate G into the right form before taking its trace. Concretely, in appendix A.1, we construct WKB eigenfunction solutions to get explicit formulas for the leading order asymptotics of the eigenfunctions. Afterwards, in appendix A.2 we use the classical Poisson summation formula and the Debye expansion to write the leading order asymptotics for G as a certain propagator, which relates the eigenfunctions to geodesics in the annulus. At that point, we use section 3.1 to show how all our constructions are quite natural and directly relate to the wave propagator appearing in [30]. Finally, we complete the proof in section 3.2 by taking traces and carrying out the method of steepest descent and stationary phase to obtain the desired asymptotic formula appearing in Proposition 2.3.

In the further analysis, we employ the summation formula,

$$\sum_{m=-l}^{l} Y_l^m(\theta, \psi) Y_l^m(\theta_0, \psi_0) = \frac{(2l+1)}{4\pi} P_l(\cos \Theta),$$

where the P_l are the Legendre polynomials, with $P_l(1) = 1$, and Θ signifies the angular epicentral distance,

$$\cos \Theta = \cos \theta \cos \theta_0 + \sin \theta \sin \theta_0 \cos(\psi - \psi_0). \tag{3.1}$$

Remark 3.1. We note that $\hat{G}$ is the kernel of the resolvent in the time-harmonic formulation. The normal mode summation becomes

$$\hat{G}(x, x_0, \omega) = \frac{1}{2\pi} \sum_{l=0}^{\infty} \sum_{n=0}^{\infty} (l + \tfrac{1}{2}) \, _n\hat{f}_l(\omega) \, \underbrace{_n\mathbf{D}_l(_n\mathbf{D}_l)_0}_{=:_nH_l} P_l(\cos \Theta), \tag{3.2}$$

explicitly showing the eigenfrequencies as simple poles (cf. (2.8)).

We abuse notation and denote

$$_nH_l = k^{-2}U(r)U(r_0)$$

in the formula for G to not treat the curl operations at first. This will not cause a risk of confusion since we will specify the exact moment that we apply the curl operators, which will be just before taking the trace in subsection 3.2.

In appendix A, we derive an asymptotic expansion of the solutions to (2.10) and use several special function identities. It is shown there the following proposition.

Proposition 3.2. *Asymptotically as* $\omega \to \infty$, *we have the following relation*

$$\hat{G}(x, x_0, \omega) \simeq \frac{1}{4\pi} \sum_{s=1,3,5,\dots} (-)^{(s-1)/2} (rr_0 c(r)c(r_0))^{-1} (2\pi\rho(r)\rho(r_0)\sin \Theta)^{-1/2}$$

$$\int (\beta(r;p)\beta(r_0;p))^{-1/2} \sum_{i=1}^{\infty} \exp[-\mathrm{i}\,\omega(\tau_i(r, r_0; p) + p\Theta + (s-1)p\pi)]$$

$$\exp[\mathrm{i}(\pi/4)(2N_i - 1)](\omega p)^{-3/2}\,\mathrm{d}p$$

$$+ \frac{1}{4\pi} \sum_{s=2,4,\dots} (-)^{s/2} (rr_0 c(r)c(r_0))^{-1} (2\pi\rho(r)\rho(r_0)\sin\Theta)^{-1/2}$$

$$\int (\beta(r;p)\beta(r_0;p))^{-1/2} \sum_{i=1}^{\infty} \exp[-\mathrm{i}\,\omega(\tau_i(r,r_0;p) + p\Theta + sp\pi)]$$

$$\exp[\mathrm{i}(\pi/4)(2N_i - 1)](\omega p)^{-3/2}\,\mathrm{d}p, \quad (3.3)$$

where τ_i is defined in appendix A and the phase function is essentially $\varphi(r,r_0,\tau,p) = \omega(\tau_i(r,r_0;p) + p\Theta + sp\pi)$.

We will show later that for certain critical values of p, $\tau_i(r,r_0;p) + p\Theta + sp\pi$ represents a geodesic length.

In the next section, we review the abstract argument appearing in [30] in order to justify why we consider the above expression a wave propagator. It will also motivate the method of steepest descent and stationary phase calculation that we perform in section 3.2.

3.1. Melrose–Guillemin wave propagators

In [30], Guillemin and Melrose show how the Neumann half-wave propagator, $e^{\pm\mathrm{i}\,t\sqrt{-\Delta_g}}$, may be written as a sum of operators denoted $\tilde{V}_\pm^i(t)$, such that for a fixed t, it is a canonical graph FIO whose canonical relation is a certain billiard map in phase space that we will briefly describe. This parametrix is only valid near rays that do not glance at the boundary and such parametrices go back to Chazarain in [12].

This FIO is closely related to (3.3) as we will soon see. In order to avoid corners, they embed M in a boundaryless manifold $\tilde{M}$ and then $\tilde{V}_+^i$ is an FIO on $\tilde{M} \times M$. It maps a covector $(y,\eta) \in S^*M^o$ to the "time t" endpoint of a broken bicharacteristic that undergoes i reflections at points $x_k(y,\eta) \in \partial M$, $k = 1,\dots,i$. Precisely, let $t_i = t_i(y,\eta)$ denote the travel time along the broken bicharacteristic to the last reflection point $(x_i,\xi_i) \in \partial_+ S^*M$ and let $(x_i,\xi_i^r) \in \partial_- S^*M$ be the reflected covector pointing "inside" M. The canonical graph maps (y,η) to $\Phi^{t-t_i}(x_i,\xi_i^r) \in S^*\tilde{M}$ (here, Φ^t denotes the bicharacteristic flow described in section 2). If fewer than i reflections take place by time t or it maps outside of M, then $\tilde{V}_+^i(t)$ is microlocally smoothing at such covectors. It will be convenient to denote a covector locally in polar coordinates as $\eta = |\eta|_{g(y)}\,\hat{\eta} \in T_y^*M$. Then the corresponding conic Lagrangian may by parameterized by a phase function of the form

$$\varphi_i(t,x,y,\eta) = |\eta|_g \left(-t + S_i(x,y,\hat{\eta})\right),$$

where, essentially, S_i gives the travel time between points x,y of the broken geodesic undergoing i reflections that starts at $(y,\hat{\eta})$ (when one finds $\hat{\eta} \in S_y^*M$ that minimizes S_i).

For the following calculations, all that is necessary is that $\tilde{V}_+^i(t)$ is a canonical graph FIO and that φ_i is a phase function that locally parameterizes the conic Lagrangian associated to the canonical graph. Thus, the Schwartz kernel of $\tilde{V}_+^i(t)$ is indeed given locally (as described in [30]) by

$$(2\pi)^{-n} \int e^{\mathrm{i}(-t|\eta|_g + S_i(x,y,\eta))} a(t,x,y,\eta)\,\mathrm{d}\eta,$$

where a is a classical symbol of order $m = 0$. If we introduced spherical coordinates in η with radial variable $\omega := |\eta|_g$ and took the leading order (homogeneous of degree $m = 0$) part of the classical symbol a, then it becomes clear that (A.13) has the same form after a Fourier transform $\mathcal{F}_{\omega\to t}$ with phase function $-\omega(\tau_i(r,r_0;p) + p\Theta + (s-1)p\pi)$ corresponding to $S_i = \omega S_i(x,y,\hat{\eta})$ above. The order in ω does not match yet because we have not yet applied the curl operators nor ∂_t. The similarities become clearer as we proceed with the stationary phase calculations for both operators.

After taking the trace of the full propagator, the contribution by $\tilde{V}_+^i$ is

$$(2\pi)^{-n} \int e^{\mathrm{i}(-t|\eta|_g + S_i(x,x,\eta))} a(t,x,x,\eta) \, \mathrm{d}x \, \mathrm{d}\eta.$$

We change variables into polar coordinates by first defining a map $f \colon \mathbb{R}_\omega^+ \times S^*M \to T^*M$,

$$f(\omega, x, p) = (x, \omega p).$$

Then $f^\star |dT^*M| = \omega^{n-1} d\omega \, |dS^*M|$. Without loss of generality, since we only consider the leading order terms, we may assume that a is an m-homogeneous (in η) symbol. After a change of variables into polar coordinates (keeping in mind that a is supported in a local coordinate chart where S^*M is trivialized), the above integral is:

$$(2\pi)^{-n} \int_0^\infty e^{-\mathrm{i}\,\omega t} \omega^{n+m-1} \, \mathrm{d}\omega \int_{M \times S^{n-1}} e^{\mathrm{i}\,\omega S_i(x,x,p)} a(t,x,x,p) \, \mathrm{d}x \, \mathrm{d}p. \tag{3.4}$$

Stationary phase with a degenerate phase function

To obtain the leading order asymptotics, we apply the method of stationary phase. We denote the critical set of S_i (viewed as function on S^*M) locally as $C_{S_i} = \{(x,p) \in S^*M; d_{x,p}S = (d_x S_i(x,x,p) + d_y S_i(x,x,p), d_p S_i) = 0\}$. We may break up this set into a countable number of disjoint connected components given by the values of S_i:

$$C_{S_i} = \bigcup_{k=1}^\infty C_{T_{ik}}, \quad C_{T_{ik}} = \{(x,p) \in S^*M; S_i = T_{ik}, \ d_{x,p}S_i = 0\}.$$

By construction of the phase function, $C_{T_{ik}}$ is precisely the fixed point set of the bicharacteristic flow $\Phi^{T_{ik}}$. Let d_{ik} be the codimension of $C_{T_{ik}}$. If $C_{T_{ik}}$ intersects ∂M transversely, then we may find independent coordinates $z = z_1, \ldots, z_{2n-1}$ such that $C_{T_{ik}}$ is given by the vanishing of $z_1, \cdots, z_{d_{ik}}$ and ∂M is given by $z_{2n-1} = 0$. For notation purposes, it will be convenient to denote $z_I = (z_1, \cdots, z_{d_{ik}})$, $z_{II} = (z_{d_{ik}+1}, \cdots, z_{2n-2})$ and $\mathcal{O}$ a neighborhood where such coordinates are valid.

We assume that S_i is Bott-Morse non-degenerate in the directions normal to the critical manifold $C_{T_{ik}}$ (clean intersection hypothesis), so that after a change of coordinates still denoted by the same letters, S_i is locally given by

$$S_i(z) = T_{ik} + \frac{1}{2} \sum_{r,s=1}^{d_{ik}} \langle \partial_{z_r} \partial_{z_s} S_i(0, z_{II}, z_{2n-1}) z_r, z_s \rangle.$$

This follows precisely from the generalized Morse lemma.

One then applies the standard stationary phase argument to (3.4) in the variables z_I corresponding to normal directions of $C_{T_{ik}}$. In that argument, the standard Hessian is replaced by the normal Hessian of S_i

$$d_N^2 S_i := (\partial_{z_i} \partial_{z_j} S)_{i,j=1,\ldots,d_{ik}}|_{z_I=0},$$

which is Bott-Morse non-degenerate by assumption. For the spherically symmetric case, $\dim(C_{T_{ik}}) = 4$ which leads to a bigger singularity. In the next subsection, we apply analogous computations for taking the trace in our setting.
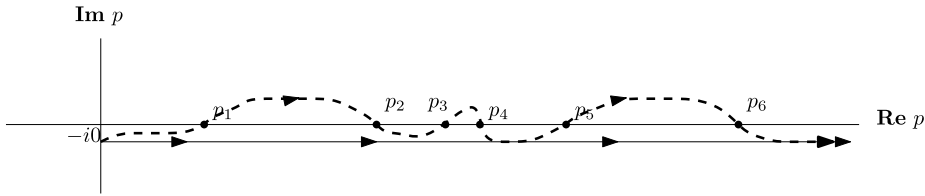
Fig. 2. As in [18, chapter 12], this is a schematic depiction of the complex p plane showing the location and orientation of the saddle points $p_k > 0$ for a given term (determined by a fixed i and s index) in $\hat{G}$. The original contour is a solid path that is parallel and just below the real axis and the deformed contour is drawn as a dashed path. This one in particular is for $i = 2$, $s = 1$ where there are six saddle points corresponding to different (broken) geodesic ray paths that connect two given points in the manifold.

3.2. Proof of Proposition 2.3

Considering (3.3), we will apply the method of steepest descent and stationary phase analogous to the argument in the previous section. The lengths of geodesics will manifest themselves as well just as in the previous section. We note that in our case, we will not take traces first, but rather apply the method of steepest descent in the variable p before taking a trace; this will not affect the analysis.

Proof of Proposition 2.3. Considering (3.3), we interchange the order of summation and integration, and invoke the method of steepest descent in the variable p. Also recall from appendix A.1 that the path of integration is beneath the real axis, while taking $\omega > 0$. We carry out the analysis for a single term, $s = 1$. For $s = 2, 4, \ldots$ we have to add $sp\pi$ to τ_i, and for $s = 3, 5, \ldots$ we have to add $(s-1)p\pi$ to τ_i, in the analysis below.

Considering

$$\varphi_{i,s=1} = \varphi_i(p) = \varphi_i(r, r_0, \Theta, p) := \tau_i(r, r_0; p) + p\Theta$$

as the phase function (for $s = 1$) and ω as a large parameter, we find (one or more) saddle points for each i, where

$$\partial_p \tau_i(r, r_0, p)|_{p=p_k} = -\Theta.$$

Later, we will consider the diagonal, setting $r_0 = r$ and $\Theta = 0$. We label the saddle points by k for each i (and s). We note that r, r_0 and Θ determine the possible values for p (given i and s) which corresponds with the number of rays connecting the receiver point with the source point (allowing conjugate points). Hence, there can be multiple saddle points for a fixed i, s, r, r_0, Θ. For $s = 1$, the rays have not completed an orbit. With $s = 3$ we begin to include multiple orbits.

We carry out a contour deformation for the method of steepest descent in the complex p plane. Initially, the contour runs from $-\infty - i0$ to $\infty - i0$, but then is deformed over the saddles as in Fig. 2. The orientation of the contour will be determined by $\operatorname{sgn} \partial_p^2 \tau_i|_{p=p_k}$ as will be described later. The appropriate contour is determined via a Taylor approximation near $\operatorname{Im} p = 0$. As in [18, chapter 12], near $p \in \mathbb{R}$, one has

$$\operatorname{Im} \tau_i \approx (\operatorname{Im} p)\Theta_i$$

where Θ_i is the epicentral distance between a source and receiver. Hence, the steepest ascent/descent paths through a saddle must have $\operatorname{Im} p$ change sign along the path.

We carry out a contour deformation over the saddles and apply the method of steepest descent to obtain

$$\frac{1}{4\pi}(-)^{(s-1)/2}(rr_0c(r)c(r_0))^{-1}(2\pi\rho(r)\rho(r_0)\sin\Theta)^{-1/2}$$

$$\int(\beta(r;p)\beta(r_0;p))^{-1/2}\sum_{i=1}^{\infty}\exp[-\mathrm{i}\,\omega(\tau_i(r,r_0;p)+p\Theta+(s-1)p\pi)]$$

$$\exp[\mathrm{i}(\pi/4)(2N_i-1)](\omega p)^{-3/2}\,\mathrm{d}p$$

$$\simeq\frac{1}{4\pi}(-)^{(s-1)/2}(rr_0c(r)c(r_0))^{-1}(2\pi\rho(r)\rho(r_0)\sin\Theta)^{-1/2}$$

$$\sum_i\sum_k\left[\omega^{-2}p^{-3/2}(\beta(r;.)\beta(r_0;.))^{-1/2}\left|\partial_p^2\tau_i(r,r_0;.)\right|^{-1/2}\right]_{p=p_k}$$

$$e^{-\mathrm{i}\,\omega T_{ik}+\mathrm{i}\,M_{ik}(\pi/2)},$$

where

$$T_{ik}=T_{s;ik}(r,r_0,\Theta)=\tau_i(r,r_0;p_k)+p_k\Delta_s,$$
$$M_{ik}=N_i-\tfrac{1}{2}(1-\mathrm{sgn}\,\partial_p^2\tau_i|_{p=p_k}),$$

in which

$$\Delta_s=\begin{cases}\Theta+(s-1)\pi & \text{if } s \text{ is odd}\\ -\Theta+s\pi & \text{if } s \text{ is even.}\end{cases}$$

The M_{ik} contribute to the KMAH indices, while the T_{ik} represent geodesic lengths or travel times. The orientation of the contour (after deformation) in the neighborhood of p_k is determined by $\mathrm{sgn}\,\partial_p^2\tau_i|_{p=p_k}$. We note that

- $M_{ik}=M_{s;ik}(r,r_0,\Theta)$ for multi-orbit waves ($s=3,4,\ldots$) includes polar phase shifts produced by any angular passages through $\Theta=0$ or $\Theta=\pi$ as well;
- if r lies in a caustic, the asymptotic analysis needs to be adapted in the usual way.

One must be careful with the term $\partial_p^2\tau_i|_{p=p_k}$ since the phase may be degenerate and this term could vanish. However, p_k depends on r,r_0,Θ and when we take the trace so that $r=r_0$ and $\Theta=0$, this term will be non-zero due to our periodic conjugacy assumption (Definition 4.2). This holds true even for those p corresponding to grazing rays.

Next, we apply both curl operations $[-\hat\theta(\sin\theta)^{-1}\partial_\psi+\hat\psi\partial_\theta]$, $[-\hat\theta_0(\sin\theta_0)^{-1}\partial_{\psi_0}+\hat\psi_0\partial_{\theta_0}]$ to each term in the sum above and then tensor the vectors together in order to obtain a sum of 2-tensor fields. This will give us the actual normal mode summation of (3.2). Since we are interested in only the leading order asymptotics in ω, we need only consider these operations to the term $\exp[-\mathrm{i}\,\omega T_{ik}]$ which gives $(\omega p_k)^2\mathbf{D}(\theta,\psi,\theta_0,\psi_0)$ where $\mathbf{D}$ is a 2-tensor field in the angular variables. We also apply an inverse Fourier transform followed by ∂_t to the formula above (since we are interested in the cosine propagator ∂_tG) to obtain to leading order

$$\simeq\frac{1}{4\pi}(-)^{(s-1)/2}(rr_0c(r)c(r_0))^{-1}(2\pi\rho(r)\rho(r_0)\sin\Theta)^{-1/2}$$

$$\sum_i\sum_k\left[p^{1/2}(\beta(r;.)\beta(r_0;.))^{-1/2}\left|\partial_p^2\tau_i(r,r_0;.)\right|^{-1/2}\right]_{p=p_k}\mathbf{D}(\theta,\psi,\theta_0,\psi_0)$$

$$\frac{1}{2\pi} \int\limits_0^\infty \mathrm{i}\,\omega \exp[-\mathrm{i}\,\omega(T_{ik} - t) + \mathrm{i}\,M_{ik}(\pi/2)]\,\mathrm{d}\omega.$$

It will now be convenient to denote the above quantity by $F^{(1)}$ if s is even and $F^{(2)}$ if s is odd.

Thus, we have a sum of kernels of FIOs associated with the wave propagator. (Here, the summation over i, k signifies the summation over (broken) geodesics while the summation over s signifies the number of orbits, traveling clockwise or counterclockwise.)

Since we will need to restrict $\partial_t G$ to the diagonal ($\Theta = 0$, $r' = r$), we must be careful with the $(\sin \Theta)^{-1/2}$ term. Fortunately, this term comes from the asymptotics of $Q^{(1)}_{\omega p - 1/2}$ and $Q^{(2)}_{\omega p - 1/2}$ (defined and proved in section A.2.1) which merely represent the two different directions of one particular geodesic. Of course, a periodic broken orbit has the same period no matter which direction one travels and in the trace formula, all orbits of a particular period are combined. Hence, we can combine $Q^{(1)}_{\omega p - 1/2}$ and $Q^{(2)}_{\omega p - 1/2}$ near $\Theta = 0$, which is a logarithmic singularity that cancels when both of the functions are added together, and this will not affect the trace formula.

Asymptotically, as $\Theta \to 0$ one has $Q^{(1)}_{\omega p_k - 1/2}(\cos(\Theta)) + Q^{(2)}_{\omega p_k - 1/2}(\cos(\Theta)) \simeq 1$ and so we may combine these terms at $\Theta = 0$ to obtain

$$\simeq -\frac{\pi}{(2\pi i)^{3/2}}(-)^{(s-1)/2}(rr_0 c(r)c(r_0))^{-1}(\rho(r)\rho(r_0))^{-1/2}$$

$$\sum_i \sum_k \left[p(\beta(r;.)\beta(r_0;.))^{-1/2} \left|\partial_p^2 \tau_i(r, r_0; .)\right|^{-1/2} \right]_{p=p_k} \mathbf{D}(\theta, \psi, \theta_0, \psi_0)$$

$$\frac{1}{2\pi} \int\limits_0^\infty i\omega^{3/2} \exp[-\mathrm{i}\,\omega(T_{ik} - t) + \mathrm{i}\,M_{ik}(\pi/2)]\,\mathrm{d}\omega,$$

as $\Theta \to 0$.

It will be convenient to denote

$$A_{s;ik}(r, r_0, \Theta) = (-)^{(s-1)/2}(rr_0 c(r)c(r_0))^{-1}(\rho(r)\rho(r_0))^{-1/2}$$

$$\cdot \left[p(\beta(r;.)\beta(r_0;.))^{-1/2} \left|\partial_p^2 \tau_i(r, r_0; .)\right|^{-1/2} \right]_{p=p_k} D(\theta, \psi, \theta_0, \psi_0)$$

where now D is a scalar function, defined as the inner product of the two vector fields that make up $\mathbf{D}$. One may check using l'Hospital's rule and equation (3.1) that $D|_{\theta=\theta_0, \psi=\psi_0} = 2$.

Next, we take the trace of $\partial_t G$ by restricting to $(r = r_0, \Theta = 0)$ and integrating. The phase function on the diagonal is $T_{ik} = \tau_i(r, r, p_k) + \pi(s - 1)p_k$ and we apply stationary phase in the variables r, θ, ψ with large parameter ω. Since one has $\partial_p T_i(r, r, p) = 0$ at $p = p_k$, the critical points occur precisely when

$$\partial_r T_i(r, r, p) + \partial_{r_0} T_i(r, r, p) = 0, \qquad \partial_p T_i(r, r, p) = 0.$$

After a quick calculation, the first condition forces T_i to be independent of r. Also, we showed that for geodesics with turning points, $U = O(\omega^{-\infty})$ when $r < R^\star$. Finally, using the inverse Fourier transform,

$$\int\limits_0^\infty \exp[i\omega(t - T)]\omega^{3/2}\,\mathrm{d}\omega = c_d(t - T + \mathrm{i}\,0)^{-5/2}, \quad \text{with } c_d \text{ a constant.}$$

Setting $R_{ik} = \{(r, \theta, \psi); r \in [\tilde{R}, 1], \partial_r T_i(r, r, p_k) + \partial_{r_0} T_i(r, r, p_k) = 0\}$ where $\tilde{R} = R^\star$ or R depending on p_k, we have shown that in fact T_{ik} remains constant over R_{ik} so that only certain i are allowable. We find that modulo terms of lower order in ω, the trace microlocally corresponds with [39]

$$\mathrm{Re} \sum_s \sum_i \sum_k \left(\frac{1}{2\pi\,\mathrm{i}}\right)^{3/2} (t - T_{s;ik} + \mathrm{i}\,0)^{-5/2}\,\mathrm{i}^{M_{ik}}\,c_d \frac{1}{2} \int\limits_{R_{ik}} A_{s;ik}(r, r, 0)\rho r^2 \sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\psi$$

and we use (A.2), (A.3). Here,

$$T_{s;ik} = T_{s;ik}(r, r; 0) = \tau_i(r, r; p_k) + \begin{cases} p_k(s-1)\pi \text{ if } s \text{ odd} \\ p_k s\pi \text{ if } s \text{ even} \end{cases}$$

is independent of r. We note that p_k exists only for i, and s, sufficiently large, which reflects the geometrical quantization.

From this expression, it is clear that the singular support of the trace consists of the travel times of periodic geodesics.

Remark 3.3. It is now apparent how the above formula relates to the trace formula in [30]. A term above for a certain i, s, k index corresponds to the trace of $\tilde{V}_i^\pm$ integrated over a critical manifold $C_{T_{ik}}$ (which in our case is R_{ik}) as described in section 3.1. In both cases, the index i is used to keep track of the number of intersections of the broken ray with the boundary while the k index specifies a particular periodic ray and period.

We further simplify the above formula, that is, the integral involving $A_{s;ik}$. First, since T_{ik} is independent of r, then so is $\tau_i(r, r; p) = \tau_i(p)$. Thus, we may pull $\partial_p^2 \tau_i$ out of the integral involving $A_{s;ik}$ precisely because we are integrating over a closed orbit:

$$\int\limits_{R_{ik}} A_{s;ik}(r, r, 0)\rho r^2 \sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\psi$$

$$= (-)^{(s-1)/2} \left| p_k^{-2} \partial_p^2 \tau_i(p_k) \right|^{-1/2} \int\limits_{R_{ik}} \frac{1}{c^2 \beta(r, p_k)} 2 \sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\psi.$$

Note that the epicentral distance denoted $\alpha_i(p)$ of the ray is $\alpha_i(p) = \partial_p \tau_i(p)$. We recall that the travel time T for a piece of a geodesic from r_0 to r is

$$T = \int\limits_{r_0}^{r} \frac{\mathrm{d}r'}{c^2 \beta(r', p_k)}\,\mathrm{d}r'.$$

Hence, denoting $T_{ik}^\sharp$ as the primitive period of the geodesic, we obtain

$$\int\limits_{R_{ik}} \frac{1}{c^2 \beta(r, p_k)} 2 \sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\psi = \frac{T_{ik}^\sharp}{N_{ik}} \int\limits_0^{2\pi} \int\limits_0^{\pi} 2 \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\psi = \frac{T_{ik}^\sharp}{\pi N_{ik}} \left| SO(3) \right|,$$

where N_{ik} is the number of geodesic segments from $\tilde{R}$ to 1 along the primitive orbit with length $T_{ik}^\sharp$, and $|SO(3)|$ is the volume of $SO(3)$ under a Haar measure [15].

Substituting these calculations, the leading order term in the trace formula is

$$\operatorname{Re} \sum_{s} \sum_{i} \sum_{k} \left(\frac{1}{2\pi \mathrm{i}} \right)^{3/2} (t - T_{s;ik} + \mathrm{i}\,0)^{-5/2}\, \mathrm{i}^{M_{ik}+s-1}\, c_d \tag{3.5}$$

$$\cdot \frac{T_{ik}^{\sharp}}{N_{ik}} \left| p_k^{-2} \partial_p \alpha_i(p_k) \right|^{-1/2} \frac{1}{2\pi}\, |SO(3)|. \quad \square$$

In the next section, we use the trace formula to prove our main spectral rigidity theorems stated in the introduction.

4. Proof of spectral rigidity

In this section we will prove that the length spectrum is rigid. By the trace formula this will imply that the spectrum of the Laplace–Beltrami operator is rigid.

4.1. Conjugacy conditions

We have two definitions taken directly from [21, section 1]. The following condition will be convenient:

Definition 4.1. We say that a $C^{1,1}$ sound speed c satisfies the countable conjugacy condition if there are only countably many radii $r \in (R, 1)$ so that the endpoints of the corresponding maximal geodesic $\gamma(r)$ are conjugate along that geodesic. In other words, the sound speed satisfies the countable conjugacy condition if every boundary point is conjugate to countably many boundary points modulo rotational symmetry.

Assuming this condition will eventually imply that the length spectrum is countable. Throughout this paper "countable" includes empty and finite sets, but for the sake of brevity we shall not write "at most countable".

The next condition is directly related to the clean intersection property discussed earlier. We will return to this condition in section 4.4.

Definition 4.2. We say that the radial wave speed c satisfies the *periodic conjugacy condition* if $\alpha(r) \in \pi\mathbb{Q}$ implies $\alpha'(r) \neq 0$. Restating geometrically, this means that if a broken ray is periodic, then the endpoints of a geodesic segment are not conjugate along the segment.

Remark 4.3. Consider a periodic broken ray of radius r. It satisfies the clean intersection property mentioned in section 2.5 if and only if either $\alpha'(r) \neq 0$ (leading to $\dim(C_T) = 4$) or α' vanishes in a neighborhood of r (leading to $\dim(C_T) = 5$).

If the second option is true, it will fail at each endpoint of the maximal interval on which α' vanishes. Since $\lim_{r \to 1} \alpha(r) = 0$ when the boundary is strictly convex, such an endpoint exists. Therefore, assuming the Herglotz condition, the clean intersection property for all periodic broken rays is equivalent with the periodic conjugacy condition of Definition 4.2. We point out that the function α is not well defined and the dimension of the fixed point set can be 3 if the Herglotz condition is violated.

4.2. Conditions for periodicity

A geodesic can be extended into a broken ray. The geodesic segments of a broken ray are rotations of each other. It is easy to see that the broken ray corresponding to the geodesic $\gamma(r)$ is periodic if and only if $\alpha(r) \in \pi\mathbb{Q}$. We want to understand the set of radii $r \in (R, 1)$ for which this is the case.

Lemma 4.4. *Let $P \subset (R,1)$ be the set of radii for which the corresponding broken ray is periodic. Let $C \subset (R,1)$ be the set of radii r for which the endpoints of the geodesic $\gamma(r)$ are conjugate along $\gamma(r)$.*

- *In fact $C = \{r \in (R,1); \alpha'(r) = 0\}$.*
- *If C has empty interior, then P is dense.*

Proof. The radius $r \in (R,1)$ parametrizes a family of geodesics. By rescaling the speed (from the originally assumed unit speed), we may assume that all geodesics are parametrized by $[-1,1]$. Differentiating the geodesic $\gamma(r)$ with respect to r gives a non-trivial Jacobi field, a variation of a geodesic $\gamma(r)$. The value of the Jacobi field at the endpoints of the geodesic describes the movement of the endpoints of the geodesic under variation. On the other hand, $\alpha(r)$ gives the endpoint of the geodesic. Therefore the tangential component of the Jacobi field at the endpoint is $\alpha'(r)$. It follows from the reparametrization that the component normal to the boundary vanishes. Thus if $\alpha'(r) = 0$, then the endpoints of $\gamma(r)$ are conjugate along $\gamma(r)$.

On the other hand, if the endpoints are conjugate, then there is a non-trivial Jacobi field vanishing at the endpoints. In dimension two there can only be a one-dimensional space of such Jacobi fields, so the Jacobi field in question must be symmetric in the time parameter of the geodesic. Then we can identify the Jacobi field as corresponding to a variation of the parameter r. Combining this with the previous observation shows that

$$r \in C \iff \alpha'(r) = 0.$$

This proves the first claim.

Let $R < a < b < 1$. To prove the second claim, it suffices to produce $r \in (a,b)$ so that $\alpha(r) \in \pi\mathbb{Q}$; see the discussion right before the statement of this lemma. For a contradiction, assume that $\alpha(r) \notin \pi\mathbb{Q}$ for all $r \in (a,b)$. Since α is continuous, this implies that α is constant on (a,b). Thus α' vanishes on (a,b), so C has interior – a contradiction. $\square$

We will state an important lemma from [21].

Lemma 4.5. *([21, proposition 28]) If the sound speed $c \in C^{1,1}$ satisfies the countable conjugacy condition (Definition 4.1) and the Herglotz condition, then C and P are countable, C is closed and P is dense in $(R,1)$.*

4.3. Length spectral rigidity

The length spectrum of a manifold M with boundary is the set of lengths of all periodic broken rays on M. If M is a spherically symmetric manifold as described above, we may choose whether or not we include the rays that reflect on the inner boundary $r = R$. If the radial sound speed is c, we denote the length spectrum without these interior reflections by $\mathrm{lsp}(c)$ and the one with these reflections by $\mathrm{lsp}'(c)$. If the inner radius is zero ($R = 0$), the manifold is essentially a ball and the two kinds of length spectra coincide.

For clarity, we state the following three length spectral rigidity theorems separately.

Theorem 4.6. *Let $B = \bar{B}(0,1) \setminus \bar{B}(0,R) \subset \mathbb{R}^n$, $n \geq 2$ and $R \geq 0$, be an annulus (or a ball if $R = 0$). Fix $\varepsilon > 0$ and let c_τ, $\tau \in (-\varepsilon, \varepsilon)$, be a $C^{1,1}$ function $[R,1] \to (0,\infty)$ satisfying the Herglotz condition and the countable conjugacy condition and depending C^1-smoothly on the parameter τ. If $R = 0$, we assume $c'_\tau(0) = 0$. If $\mathrm{lsp}(c_\tau) = \mathrm{lsp}(c_0)$ for all $\tau \in (-\varepsilon, \varepsilon)$, then $c_\tau = c_0$ for all $\tau \in (-\varepsilon, \varepsilon)$.*

Theorem 4.7. *Let $B = \bar{B}(0,1) \setminus B(0,R) \subset \mathbb{R}^n$, $n \geq 2$ and $R > 0$, be an annulus. Fix $\varepsilon > 0$ and let c_τ, $\tau \in (-\varepsilon, \varepsilon)$, be a $C^{1,1}$ function $[R,1] \to (0,\infty)$ satisfying the Herglotz condition and the countable conjugacy*

condition and depending C^1-smoothly on the parameter τ. If $\mathrm{lsp}'(c_\tau) = \mathrm{lsp}'(c_0)$ for all $\tau \in (-\varepsilon, \varepsilon)$, then $c_\tau = c_0$ for all $\tau \in (-\varepsilon, \varepsilon)$.

Notice that dimension is irrelevant for the statements; if the sound speed is fixed, the length spectrum is independent of dimension.

The following theorem states that the same rigidity result is true for any finite disjoint union of manifolds of the types given in Theorems 4.6 and 4.7.

Theorem 4.8. *Let N and N' be non-negative integers so that $N + N' \geq 1$. Let $R_1, R_2, \ldots, R_N \in [0, 1)$ and $R'_1, R'_2, \ldots, R'_{N'} \in (0, 1)$ be any numbers. Let $n_1, n_2, \ldots, n_N$ and $n'_1, n'_2, \ldots, n'_{N'}$ be integers, each of them at least 2. Fix $\varepsilon > 0$ and let $c_{i,\tau} \colon [R_i, 1] \to (0, \infty)$ for $i = 1, \ldots, N$ and $c'_{i,\tau} \colon [R'_i, 1] \to (0, \infty)$ for $i = 1, \ldots, N'$ be $C^{1,1}$ functions satisfying the Herglotz condition and the countable conjugacy condition and depending C^1-smoothly on the parameter τ. For every i such that $R_i = 0$, we assume $\frac{\mathrm{d}}{\mathrm{d}r} c_{i,\tau}(r)\big|_{r=0} = 0$. If the set*

$$\bigcup_{i=1}^{N} \mathrm{lsp}(c_{i,\tau}) \cup \bigcup_{i=1}^{N'} \mathrm{lsp}'(c'_{i,\tau})$$

is the same for all $\tau \in (-\varepsilon, \varepsilon)$, then every sound speed $c_{i,\tau}$ and $c'_{i,\tau}$ is independent of the parameter τ.

Remark 4.9. It does not matter whether for every periodic broken ray only the primitive period is included in the length spectrum, or of its all integer multiples. The proofs of the three theorems above are the same in both cases.

Remark 4.10. We emphasize that Theorems 4.6, 4.7, and 4.8 work irrespective of the multiplicity of the length spectrum. We did need a multiplicity assumption for the trace formula but not for these results on length spectral rigidity. Heuristically, the proof argues as follows: If there was a nontrivial change of the radial velocity profile, then at least one stable periodic broken ray (of which there are many) would change length when varying the parameter τ. Therefore any such change would induce a change in the length spectrum as a function of τ, even if the ray under study does not have a unique length.

Even more might be true, and we propose the following conjecture.

Conjecture 4.11. *Under certain geometric hypotheses, a spherically symmetric manifold is uniquely determined by its length spectrum.*

A verification of the conjecture would imply that such a manifold is uniquely determined by its spectrum under some geometric assumptions.

Remark 4.12. Theorem 4.8 may seem like an unnecessary generalization of the two preceding theorems, but it has geophysical significance. Consider a spherically symmetric model of the earth. It essentially consists of three different parts, an inner ball and two nested annuli. The full length spectrum of the Earth is the set of all periodic orbits for the different polarized waves. The statement in Theorem 4.8 (with $N = 2$ and $N' = 1$) is, however, incomplete in the sense that the coupling between different polarizations and transmission at boundaries are ignored. However, this is the best toy model for which rigidity is currently known.

Radial symmetry is an excellent approximation of the earth or planets in general. This symmetry is not exact, and unfortunately our method requires precise symmetry. Assuming radial symmetry is not merely a matter of technical convenience, but a truly necessary assumption. A key ingredient in the proof is that many periodic broken rays are stable under deformations of the metric. In spherical symmetry the broken

rays are only stable under deformations that preserve the symmetry, otherwise they are typically unstable and the proof falls apart.

The countable conjugacy condition should be a generic property of sound speeds. It is a technical assumption we do not like to make, but it does not hinder the relevance for our planetary model.

The Herglotz condition is crucial for the geometry of the problem. Without it the manifold would trap some geodesics inside and the geometry of broken rays would be very different. In the commonly used Preliminary Reference Earth Model (PREM) both pressure and shear wave speeds satisfy the Herglotz condition piecewise. Due to the layered structure of the Earth both wave speeds have jumps, whereas our result assumes $C^{1,1}$ regularity.

4.3.1. No inner reflections

In this subsection we prove Theorem 4.6. All the lemmas here are stated under the assumptions of the theorem. By decreasing ε slightly we can assume that $c_\tau(1)$ is bounded away from zero and infinity without any loss of generality.

Let $P_\tau \subset (R, 1)$ be the set of radii for which the corresponding broken ray – which is unique up to rotations – is periodic with respect to the sound speed c_τ. A priori P_τ depends on τ. If $r \in P_\tau$, the corresponding broken ray has $n_\tau(r)$ reflections and winds around the origin $m_\tau(r)$ times. We choose all periodic broken rays to have minimal period, so the natural numbers $n_\tau(r)$ and $m_\tau(r)$ are coprime. If $\alpha_\tau(r)$ is the angle we defined earlier (previously without the dependence on τ), we have the identity

$$\pi m_\tau(r) = n_\tau(r)\alpha_\tau(r).$$

We denote the length of the periodic broken ray with radius r by $\ell_\tau(r)$. Simple geometrical considerations show that $\ell_\tau(r) = 2n_\tau(r)L_\tau(r)$, where $L_\tau(r)$ is defined like $L(r)$ in (2.1). We denote $\rho_\tau(r) = r/c_\tau(r)$. The Herglotz condition states that $\rho'_\tau(r) > 0$.

Lemma 4.13. *Assume $R > 0$. There is a constant $C > 1$ so that*

- $\frac{1}{C} < c_\tau(r) < C$,
- $\frac{1}{C}(s - r) < \rho_\tau(s) - \rho_\tau(r) < C(s - r)$,
- $\alpha_\tau(r) < C$ *and*
- $L_\tau(r) < C$

for all $\tau \in (-\varepsilon, \varepsilon)$ and $r, s \in (R, 1)$ with $s > r$.

Proof. It follows from the Herglotz condition that $r/c_\tau(r) \leq 1/c_\tau(1)$, whence $c_\tau(r) \geq Rc_\tau(1)$. Since $c_\tau(1)$ is uniformly bounded from below, the functions c_τ are all uniformly bounded from below. We assumed the sound speeds to be uniformly bounded from above in the theorem, so for some constant $C > 1$ we have $\frac{1}{C} < c_\tau(r) < C$ for all τ and r.

We write $LHS \lesssim RHS$ if there is a constant C independent of τ, r and s so that $LHS < C \cdot RHS$. By $LHS \approx RHS$ we mean $LHS \lesssim RHS \lesssim LHS$.

Assume $s > r$. Since c_τ is $C^{1,1}$ and satisfies the Herglotz condition, we have $0 < \rho_\tau(s) - \rho_\tau(r) \lesssim s - r$. On the other hand the minimum of ρ'_τ depends continuously on τ and is always positive, so we have also $s - r \lesssim \rho_\tau(s) - \rho_\tau(r)$. Thus $\rho_\tau(s) - \rho_\tau(r) \approx s - r$.

It follows from the previous observation that

$$\left(1 - \left(\frac{\rho_\tau(r)}{\rho_\tau(s)}\right)^2\right)^{-1/2} \approx (s - r)^{-1/2}.$$

Combining this with equations (2.2) and (2.1) gives the desired uniform estimates for α_τ and L_τ. $\quad\square$

We will state several facts that follow from the geometry of the problem. First, if we let $R = 0$, then this corresponds to a geodesic that tends to the diameter of a ball and so

$$\lim_{r\to 0} \alpha_\tau(r) = \frac{\pi}{2} \text{ and } \lim_{r\to 0} L_\tau(r) = \int\limits_0^1 c_\tau(r)^{-1} \mathrm{d}r.$$

Next, note that the length spectrum $\mathrm{lsp}(c_\tau) = \{\ell_\tau(r); r \in P_\tau\}$ is countable due to the countability of P_τ (Lemma 4.5). We denote by S_τ the set of radii $r \in (R, 1)$ for which $\alpha(r) \in \pi\mathbb{Q}$ and $\alpha'(r) \neq 0$. Radii in this set will correspond to stable broken rays as we shall see in below. The set S_τ is countable and dense in $(R, 1)$. This is because P_τ is dense and countable, the set C_τ of zeros of α'_τ is closed and countable, and the observation that $S_\tau = P_\tau \setminus C_\tau$.

Let us observe that periodic broken rays are stable under variations of the radial sound speed. Periodic broken rays on a highly symmetric manifold are typically not stable under all variations of the metric, but we are only looking at variations that preserve the symmetry. For every $r \in S_0$ there is $\delta \in (0, \varepsilon)$ a C^1 function

$$\varphi \colon (-\delta, \delta) \to (R, 1) \tag{4.1}$$

so that

- $\varphi(0) = r$,
- $\alpha_\tau(\varphi(\tau)) = \alpha_0(r)$ (and thus $\varphi(\tau) \in P_\tau$),
- $n_\tau(\varphi(\tau)) = n_0(r)$ and
- $m_\tau(\varphi(\tau)) = m_0(r)$

for all $\tau \in (\delta, \delta)$. In particular, $\tau \mapsto \ell_\tau(\varphi(\tau))$ is differentiable.

This follows from the implicit function theorem since for a fixed $r \in S_0$, we know that $\alpha'_0(r) \neq 0$, so there is a C^1 function φ defined near zero so that $\alpha_\tau(\varphi(\tau)) = \alpha_0(r)$ and $\varphi(0) = r$. Since $\alpha_\tau(\varphi(\tau))$ is independent of τ, so are the numbers $n_\tau(\varphi(\tau))$ and $m_\tau(\varphi(\tau))$. Differentiability of the length follows from the fact that the reflection number is constant and L is differentiable.

Finally, there are two important equations that we need. Let $\varphi \colon (-\delta, \delta) \to (R, 1)$ for any $\delta > 0$ be a C^1 function satisfying $\varphi(\tau) \in P_\tau$ for all $\tau \in (-\delta, \delta)$. Let $\gamma_0 \colon [0, T] \to M$ be a periodic broken ray with radius $\varphi(0)$. Then one can compute

$$2 \left. \frac{\mathrm{d}}{\mathrm{d}\tau} \ell_\tau(\varphi(\tau)) \right|_{\tau=0} = \int\limits_0^T \left. \frac{\mathrm{d}}{\mathrm{d}\tau} c_\tau^{-2}(\gamma_0(t)) \right|_{\tau=0} \mathrm{d}t. \tag{4.2}$$

If $f \colon M \to \mathbb{R}$ is a continuous radially symmetric function (identified as a function $f \colon (R, 1] \to \mathbb{R}$) and $r \in P_0$, then the integral of f over any periodic geodesic (with respect to sound speed c_0) of radius r is

$$2n_0(r) \int\limits_r^1 \frac{f(s)}{c(s)} \left(1 - \left(\frac{rc(s)}{sc(r)} \right)^2 \right)^{-1/2} \mathrm{d}s. \tag{4.3}$$

Lemma 4.14. *If $Af(r)$ is the function of (4.3), then the map $f \mapsto Af$ takes continuous functions to continuous functions and is injective on the space of continuous functions.*

Proof. This follows from theorems 5 and 12 (or lemma 25) of [21]. □

In the C^∞ setting relevant for the spectrum of the Laplace–Beltrami operator this was shown by Sharafutdinov [42]. With the help of all these auxiliary results, it is difficult not to prove Theorem 4.6.

Proof of Theorem 4.6. Take any radius $r \in S_0$ and let φ be the function (4.1) constructed there. We know that $\tau \mapsto \ell_\tau(\varphi(\tau))$ is differentiable, $\ell_\tau(\varphi(\tau)) \in \mathrm{lsp}(c_\tau) = \mathrm{lsp}(c_0)$ and $\mathrm{lsp}(c_0)$ has no interior. Therefore this function is constant.

By (4.2) this implies that the variation of the wave speed, $f = \frac{\mathrm{d}}{\mathrm{d}\tau} c_\tau^{-2}\big|_{\tau=0} : M \to \mathbb{R}$, integrates to zero over all periodic geodesics of radius r. This function f is radially symmetric, so we can think of it as a function $(R, 1] \to \mathbb{R}$. By (4.3) we know that

$$\int\limits_r^1 \frac{f(s)}{c(s)} \left(1 - \left(\frac{rc(s)}{sc(r)}\right)^2\right)^{-1/2} \mathrm{d}s = 0. \tag{4.4}$$

Equation (4.4) is true for a dense set of radii $r \in (R, 1)$ by our discussion of S_τ earlier, so it follows from Lemma 4.14 that in fact f vanishes identically.

We have found that $\frac{\mathrm{d}}{\mathrm{d}\tau} c_\tau = 0$ at $\tau = 0$. The choice $\tau = 0$ was in no way important to this argument, so in fact $\frac{\mathrm{d}}{\mathrm{d}\tau} c_\tau = 0$ for all $\tau \in (-\varepsilon, \varepsilon)$. This means that all sound speeds c_τ indeed coincide. □

A key step in the reasoning in the preceding proof can be stated as follows: A radially symmetric function is uniquely determined by its integrals over all periodic broken rays. There is even a reconstruction formula for this problem: [21, Remark 27]. Radial symmetry is important, as a general smooth function is not uniquely determined. Only the even part of the function can be recovered efficiently. This is in sharp contrast to the case of geodesic X-ray tomography on such manifolds where the ray transform has no kernel. See [21] for details.

4.3.2. Inner reflections included

In this subsection we will prove Theorems 4.7 and 4.8.

The only ingredient we need to add to the proof of Theorem 4.6 is the following lemma. We will prove the lemma after showing how it completes the proofs of the theorems.

Lemma 4.15. *Let $M = \bar{B}(0,1) \setminus B(0,R) \subset \mathbb{R}^n$, $R > 1$ and $n \geq 2$, be an annulus. Equip M with a radially symmetric $C^{1,1}$ sound speed satisfying the Herglotz condition. Then the set of all lengths of periodic broken rays that reflect on the inner boundary $r = R$ is countable.*

We note that no assumption was made on conjugate points in the lemma.

Proof of Theorem 4.7. We denote the set of all lengths of periodic broken rays with respect to sound speed c_τ that reflect on the inner boundary by H_τ. Then we have $\mathrm{lsp}'(c_\tau) = \mathrm{lsp}(c_\tau) \cup H_\tau$. As in the proof of Theorem 4.6, we have a dense set of radii $r \in (R, 1)$ for which there is a family of corresponding periodic broken rays varying continuously in τ; this set was denoted by S_0. Since $\mathrm{lsp}'(c_\tau)$ is independent of τ and each set H_τ is countable, the lengths of the periodic broken rays in this family must in fact be independent of τ. The rest of the proof can be concluded as that of Theorem 4.6. We only need the integrals of the variations of the sound speed over broken rays that do not hit the inner boundary. □

Proof of Theorem 4.8. We denote the individual Riemannian manifolds in question by $M_1, \ldots, M_N$ and $M'_1, \ldots, M'_{N'}$, equipped with their respective sound speeds. Each one of them has a countable length spectrum, so the length spectrum of the whole system is still countable. We can then use the argument presented

in the proof of Theorem 4.7 to conclude for each manifold separately that the sound speed has to be independent of the parameter τ. □

Proof of Lemma 4.15. Consider a geodesic joining the two boundaries; a periodic broken ray of the kind we need to study is a finite union of rotations and reflections of such a geodesic. We parametrize this geodesic with arc length starting from the inner boundary. Due to symmetry the geodesic is confined to a two-dimensional plane, and in this plane we may use polar coordinates. In these coordinates the geodesic is $[0, T] \ni t \mapsto (r(t), \theta(t))$, with $r(0) = R$ and $r(T) = 1$. We denote $\rho(r) = r/c(r)$.

Let the angle between the geodesic and the radially outward pointing normal vector be ω. (The metric is conformally Euclidean so we may use the Riemannian metric or the Euclidean metric to measure angles at a point.) We assume for the time being that $\omega \in (0, \frac{\pi}{2})$. Since the geodesic has unit speed, we have $\cos \omega = \rho(R)\theta'(0)$. With this information it is easy to see that the constant value of the angular momentum $\rho(r(t))^2\theta'(t)$ is $\rho(R)\cos\omega$.

Using unit speed and the conservation of angular momentum one easily finds that the change in the angular coordinate over the geodesic is

$$\theta(T) - \theta(0) = \int_R^1 \frac{\rho(R)\cos\omega}{c(r)\rho(r)^2} \left(1 - \left(\frac{\rho(R)\cos\omega}{\rho(r)}\right)^2\right)^{-1/2} \mathrm{d}r.$$

This angular difference depends on ω, but it is most convenient to think of it as a function of $z := \rho(R)\cos\omega$ (the angular momentum). We denote this angular difference by $\beta(z)$. Its geometrical meaning is the same as that of $\alpha(r)$ defined earlier, but we use a different letter to avoid confusion.

Since

$$\beta(z) = \int_R^1 \frac{z}{c(r)\rho(r)^2} \left(1 - \left(\frac{z}{\rho(r)}\right)^2\right)^{-1/2} \mathrm{d}r, \tag{4.5}$$

an easy calculation gives

$$\beta'(z) = \int_R^1 \frac{1}{c(r)\rho(r)^2} \left(1 - \left(\frac{z}{\rho(r)}\right)^2\right)^{-3/2} \mathrm{d}r.$$

This derivative is positive, so β is in fact a homeomorphism from $[0, 1]$ to its image; the limits at 0 and 1 can be checked to be finite.

We may exclude radial geodesics. For non-radial geodesics the integral of (4.5) is non-singular and differentiation under the integral sign is simple. For the similar result corresponding to the diving waves (the function α), the derivative is more complicated; cf. [21, proposition 15].

The broken ray corresponding to angular momentum z is periodic if and only if $\beta(z) \in \pi\mathbb{Q}$. Since β is a homeomorphism between intervals, the set of angular momenta z corresponding to periodicity is countable. Therefore the set of corresponding lengths is also countable. □

4.4. Spectral rigidity

We are now ready to prove Theorem 1.3. The proof is trivial at this point, but we record it explicitly for completeness.

Proof of Theorem 1.3. Let us first consider the simplest case where c satisfies the periodic conjugacy condition.

The trace of the Green's function is determined by spec(c) through Proposition 2.3. Since the trace as a function of $t \in \mathbb{R}$ is singular precisely at the length spectrum, spec(c) determines lsp$'(c)$. Then rigidity of the spectrum spec(c) follows from that of lsp$'(c)$; see Theorem 4.7.

Let us then drop the periodic conjugacy condition. The Neumann spectrum spec(c) still determines the trace of the Green's function. As pointed out in Remark 2.6, the singularities determine the part of the length spectrum corresponding to periodic broken rays satisfying the clean intersection property. Under the Herglotz condition and the countable conjugacy condition a periodic broken ray of radius r satisfies the clean intersection property if and only if $\alpha'(r) \neq 0$; see Remark 4.3. In the notation of Lemma 4.4 and denoting the length of a geodesic of radius r by $\ell(r)$, the problematic primitive lengths are precisely $\ell(r)$ for $r \in C \cap P$. Since we assumed the clean intersection hypothesis, none of the lengths $\ell(r)$ for $r \in P \setminus C$ coincide with the problematic ones. Access to all radii $r \in P \setminus C$ is sufficient for the proof of length spectral rigidity. Notice that C is exactly the same set that corresponds to possibly unstable periodic broken rays, and this data was ignored in the proof of length spectral rigidity anyway. $\square$

Appendix A. Eigenfunction asymptotics

In this appendix, we analyze the asymptotic behavior of the normal mode summation $\hat{G}(x, x_0, \omega)$ in order to relate it to a certain wave propagator that we will analyze using the method of steepest descent. This connection between $\hat{G}(x, x_0, \omega)$ and the wave propagator will come from doing an asymptotic analysis of the WKB eigenfunctions combined with the Debye expansion in appendix A.2.2 applied to the summation of the eigenfunctions. We reiterate that a careful study of the eigenfunctions allows us to obtain a precise trace formula even with non simple periodic orbits (as opposed to [30] where their formula only holds when there is no symmetry) and grazing rays at the inner boundary.

First, we look at the asymptotics of the radial WKB eigenfunctions and then in section A.2, we use them to analyze the asymptotics of the eigenfunction summation. Much of the analysis is taken from [18, chapter 12]. The main result that we need for the trace formula is (A.13).

A.1. Asymptotic analysis of eigenfunctions

We describe the radial eigenfunctions and their asymptotic expansions for general $k \in \mathbb{R}$. We then introduce the dispersion relations, $\omega_n(k)$.

A.1.1. WKB eigenfunctions
Here, we consider asymptotic solutions, V, to (2.10). Depending on p, we distinguish the following regimes:

- **Evanescent** $(1/c(1) < p < \infty)$. Here, $\beta^2(r) < 0$, and the solution is always non-oscillatory, that is, evanescent. We do not obtain eigenfunctions.
- **Diving** $(R/c(R) < p < 1/c(1))$: *Diving waves* are waves that correspond to turning rays and we construct eigenfunctions corresponding to these rays. We summarize the WKB solution of (2.10) in the vicinity of a general turning point. A turning point, $r = R^\star$, is determined by

$$\beta^2(R^\star) = 0.$$

Near a turning point, $r \approx R^\star$, and

$$\beta^2(r) \simeq q_0(r - R^\star),$$

for an q_0 determined by a Taylor expansion. Away from a turning point,

$$\beta^2 > 0 \text{ if } r \gg R^\star, \quad \beta^2 < 0 \text{ if } r \ll R^\star.$$

Matching asymptotic solutions yields

$$B \begin{cases} |\beta|^{-1/2} \exp\left(-\omega \int_r^{R^\star} |\beta| \, \mathrm{d}r\right), & r \ll R^\star \\ 2\pi^{1/2} q_0^{-1/6} \omega^{1/6} \mathrm{Ai}(-\omega^{2/3} q_0^{1/3}(r - R^\star)), & r \simeq R^\star \\ 2\beta^{-1/2} \cos\left(-\omega \int_{R^\star}^r \beta \, \mathrm{d}r - \pi/4\right), & r \gg R^\star. \end{cases}$$

From these one can obtain a uniform expansion, that is, the Langer approximation

$$V(r, \omega; p) = 2\pi^{1/2} \chi^{1/6} (-\beta^2)^{-1/4} \mathrm{Ai}(\chi^{2/3}(r)),$$

$$\chi(r) = -(3/2)\omega \int_{R^\star}^r (-\beta^2)^{1/2} \, \mathrm{d}r, \qquad\qquad (A.1)$$

valid for $r \in [R, 1]$. One obtains eigenfunctions corresponding with turning rays.

Up to leading order, where $r \gg R^\star$,

$$V = 2B\beta^{-1/2} \cos\left(\omega \int_{R^\star}^r \beta \, \mathrm{d}r' - \pi/4\right),$$

$$\partial_r V = -2\omega B\beta^{1/2} \sin\left(\omega \int_{R^\star}^r \beta \, \mathrm{d}r' - \pi/4\right).$$

Here, B is obtained from the normalization (2.6), which requires the uniformly asymptotic solution over the entire interval $[R, 1]$. Applying the Riemann–Lebesgue lemma, one obtains

$$\int_R^1 U(r)^2 \rho(r) r^2 \, \mathrm{d}r \simeq 2B^2 \int_{R^\star}^1 \beta^{-1} c^{-2} \, \mathrm{d}r. \qquad\qquad (A.2)$$

Here, $\int_{R^\star}^1 \beta^{-1} c^{-2} \, \mathrm{d}r$ can be identified with the one half-return travel time, $\frac{1}{2}T$, say.

- **Grazing** $(R = R^*)$: It is possible that for certain turning rays, $R^* = R$, in which case the Neumann boundary condition $\partial_r V|_{r=R=R^*} = 0$ must be satisfied as well. This condition will be satisfied by using the representation (A.1) near the grazing point and also introducing $\mathrm{Bi}(x)$ in addition to $\mathrm{Ai}(x)$ above so that near $r = R^*$, V is a linear combination of $\omega^{1/6} \mathrm{Ai}(-\omega^{2/3} q_0^{1/3}(r - R^\star))$ and $\omega^{1/6} \mathrm{Bi}(-\omega^{2/3} q_0^{1/3}(r - R^\star))$. So for $r \simeq R^*$, we have

$$V(r) \simeq 2\pi^{1/2} q_0^{-1/6} \omega^{1/6} [B_1 \mathrm{Ai}(-\omega^{2/3} q_0^{1/3}(r - R^\star)) + B_2 \mathrm{Bi}(-\omega^{2/3} q_0^{1/3}(r - R^\star))]$$

Then

$$\partial_r V(r)|_{r=R^*} = -2\pi^{1/2} q_0^{1/6} \omega^{-1/2} [B_1 \mathrm{Ai}'(0) + B_2 \mathrm{Bi}'(0)]$$

Setting $\partial_r V(r)|_{r=R^*} = 0$ and using $-\mathrm{Ai}'(0) = \mathrm{Bi}'(0)/\sqrt{3}$ we get $B_1 = \sqrt{3} B_2$. The constants may be calculated as before using the Riemann-Lebesgue lemma to yield that B_2 is equal to one fourth of the constant in the previous case. The details can be found in [18, chapter 12] and [38].

We also note that under the Herglotz condition, only simple tangency is possible at the inner boundary.

Lemma A.1. *Let $\gamma(t)$ be a geodesic that hits the inner boundary $r = R$ tangentially at $t = 0$. Then the tangency is simple.*

Proof. Let $p(r, \xi_r, \xi_\theta, \xi_\varphi) = c^2(\xi_r^2 + \frac{1}{r^2}\xi_\theta^2 + \frac{1}{r^2 \sin^2\theta}\xi_\varphi^2)$ be the principal symbol of $-\Delta_c$ with Hamilton vector field

$$H_p = 2c^2\xi_r\partial_r - \partial_r\left(\frac{c}{r}\right)^2\left(\xi_\theta^2 + \frac{1}{\sin^2\theta}\xi_\varphi^2\right)\partial_{\xi_r} + D(r, \theta, \partial_\theta, \partial_\varphi)$$

where $D(r, \theta, \partial_\theta, \partial_\varphi)$ is a vector field involving only angular derivatives. Next $\tilde{r} = r - R$ is a boundary defining function for the inner boundary. The assumption implies that $H_p\tilde{r} = 0$ on γ at $t = 0$ so it suffices to show that $H_p^2\tilde{r} \neq 0$ at this point. First, $H_p\tilde{r} = 2c^2\xi_r = 0$ implies $\xi_r = 0$ at this point. Then

$$H_p^2\tilde{r} = 2c^2\xi_r^2(\partial_r c^2) - \partial_r\left(\frac{c}{r}\right)^2\left(\xi_\theta^2 + \frac{1}{\sin^2\theta}\xi_\varphi^2\right)c^2 = -\partial_r\left(\frac{c}{r}\right)^2\left(\xi_\theta^2 + \frac{1}{\sin^2\theta}\xi_\varphi^2\right)c^2$$

when restricted to $\gamma(0)$. By the Herglotz condition, this last quantity is nonzero since $(\xi_\theta, \xi_\varphi) \neq (0, 0)$ and the lemma is proved. □

Remark A.2. Without the Herglotz condition, one could theoretically have rays that graze the inner boundary in a complicated way. If we were doing a parametrix construction for the wave equation, such a parametrix is not always possible and it is more complicated than the one for hyperbolic points (see [45,2]). Such complications would arise for the WKB solutions as well, but here, the Herglotz condition ensures that the ray is merely a turning ray that intersects the inner boundary tangentially with exactly second order contact.

- **Reflecting** $(0 < p < R/c(R))$: The solutions are oscillatory in the entire interval $[R, 1]$ $(\beta^2(r; p) > 0)$, correspond with reflecting rays, and are of the form

$$V = C\beta^{-1/2}\exp\left(i\omega\int_R^r \beta \, dr'\right) + D\beta^{-1/2}\exp\left(-i\omega\int_R^r \beta \, dr'\right),$$

$$\partial_r V = i\omega C\beta^{1/2}\exp\left(i\omega\int_R^r \beta \, dr'\right) - i\omega D\beta^{1/2}\exp\left(-i\omega\int_R^r \beta \, dr'\right),$$

to leading order. Imposing the Neumann condition, $T(R) = 0$, implies that $D = C$. The constant C is obtained from the normalization (2.6) using the oscillatory solution over the entire interval $[R, 1]$. Applying the Riemann–Lebesgue lemma yields

$$\int_R^1 U(r)^2\rho(r)r^2 \, dr \simeq 2C^2\int_R^1 \beta^{-1}c^{-2} \, dr. \tag{A.3}$$

Here, $\int_R^1 \beta^{-1}c^{-2} \, dr$ can be identified with the half one-return reflection travel time, $\frac{1}{2}T$, say.

A.1.2. Boundary condition and dispersion relations

We backsubstitute $p = \omega^{-1}k$ in β. Imposing the Neumann boundary condition (at the outer boundary $r = 1$) yields:

- **Diving** $(R/c(R) < p < 1/c(1))$:

$$\omega \int_{R^*}^{1} \beta(r'; \omega^{-1}k) \, dr' = \left(n + \frac{5}{4}\right)\pi \quad (n \text{ is the overtone index}). \tag{A.4}$$

- **Grazing**: Using the asymptotic of the Airy function, we have

$$\partial_r V|_{r=1} = -2\omega B_1 \beta^{1/2} \sin\left(\omega \int_{R^*}^{1} \beta(r'; \omega^{-1}k) \, dr' - \pi/4\right) + 2\omega B_2 \beta^{1/2} \cos\left(\omega \int_{R^*}^{1} \beta(r'; \omega^{-1}k) \, dr' - \pi/4\right)$$

So the Neumann condition gives

$$\tan\left(\omega \int_{R^*}^{1} \beta(r'; \omega^{-1}k) \, dr' - \pi/4\right) = -1/\sqrt{3}$$

so

$$\omega \int_{R^*}^{1} \beta(r'; \omega^{-1}k) \, dr' = \left(n + \frac{13}{12}\right)\pi \tag{A.5}$$

analogous to [38, equation (22)]. Details for this and more general situations may be found in [38].

- **Reflecting** $(0 < p < R/c(R))$:

$$\omega \int_{R}^{1} \beta(r'; \omega^{-1}k) \, dr' = (n+1)\pi \quad (n \text{ is the overtone index}). \tag{A.6}$$

Remark A.3. Note that these conditions are very close to the Einstein-Brillouin-Keller semiclassical method (EBK) used to compute eigenvalues in quantum-mechanical systems. Indeed, by introducing the semiclassical parameter $h = 1/\omega$, these become the quantization conditions for a quantum-mechanical system under certain assumptions.

All these (radial) quantization-type conditions yield solutions $_n\omega_k =: \omega_n(k)$. Using the implicit function theorem, we introduce $k_n = k_n(\omega)$ as the solution of

$$\omega - \omega_n(k_n) = 0.$$

We revisit the general relation between phase and group velocities. We have

$$c_n(k) = \frac{\omega_n(k)}{k}$$

and

$$C_n = \frac{d(c_n k)}{dk} = c_n + k \frac{dc_n}{dk}.$$

The corresponding ray parameter is given by

$$p = \frac{k}{\omega_n(k)} = \frac{1}{c_n(k)}. \tag{A.7}$$

A.2. Poisson's summation formula

In the previous section, we described the leading order asymptotics of the WKB eigenfunctions. In this section, we use these asymptotics to analyze the eigenfunction summation $\hat{G}(x, x_0, \omega)$ asymptotically and relate it to the wave propagator. First we need some properties of special functions appearing in the equation for $\hat{G}$.

A.2.1. Application of Poisson's formula

Let us first use *Poisson's formula* to rewrite $\hat{G}(x, x_0, \omega)$ in a different form. Poisson's formula is given by

$$\sum_{l=0}^{\infty} f(l + \tfrac{1}{2}) = \sum_{s=-\infty}^{\infty} (-)^s \int_0^\infty f(k) e^{-2\,\mathrm{i}\,sk\pi}\,\mathrm{d}k.$$

We apply Poisson's formula to the summation in l in (3.2) while keeping the summation in n intact. We obtain after rewriting

$$\hat{G}(x, x_0, \omega) = \frac{1}{2\pi} \sum_{s=1}^{\infty} (-)^s \int_0^\infty \left[\sum_{n=0}^{\infty} \hat{f}_n(k; \omega)\, H_n(k) \right] P_{k-1/2}(\cos\Theta) \{ e^{-2\,\mathrm{i}\,sk\pi} + e^{2\,\mathrm{i}\,sk\pi} \}\, k\,\mathrm{d}k$$

$$+ \frac{1}{2\pi} \int_0^\infty \left[\sum_{n=0}^{\infty} \hat{f}_n(k; \omega)\, H_n(k) \right] P_{k-1/2}(\cos\Theta)\, k\,\mathrm{d}k. \quad (A.8)$$

Traveling-wave Legendre functions

Let us describe the properties of the Legendre functions appearing in the formula for $\hat{G}(x, x_0, \omega)$ above that will allow us to obtain their asymptotic behavior.

There are two traveling wave Legendre functions whose asymptotic behaviors, as $|\lambda| \gg 1$, are (assuming that Θ is sufficiently far away from the endpoints of $[0, \pi]$) is [18, Chapter 12]

$$Q^{(1)}_{\lambda-1/2} \simeq \left(\frac{1}{2\pi\lambda\sin\Theta} \right)^{1/2} e^{-\mathrm{i}(\lambda\Theta - \pi/4)}, \quad (A.9)$$

$$Q^{(2)}_{\lambda-1/2} \simeq \left(\frac{1}{2\pi\lambda\sin\Theta} \right)^{1/2} e^{+\mathrm{i}(\lambda\Theta - \pi/4)},$$

upon substituting $z = \cos\Theta$. Taking into consideration the time-harmonic factor $e^{\mathrm{i}\omega t}$, it follows that $Q^{(1)}$ represents waves traveling in the direction of increasing Θ, while $Q^{(2)}$ represents waves traveling in the direction of decreasing Θ.

To distinguish the angular directions of propagation, one decomposes [18, Chapter 12]

$$P_{k-1/2}(\cos\Theta) = Q^{(1)}_{k-1/2}(\cos\Theta) + Q^{(2)}_{k-1/2}(\cos\Theta). \quad (A.10)$$

In preparation of the application of the method of steepest descent, these Legendre functions can be analytically extended from k real positive to real negative [18, Chapter 12]. Substituting (A.10) into (A.8), after a brief computation, we obtain [18, Chapter 12]

$$\hat{G}(x, x_0, \omega) = \frac{1}{2\pi} \sum_{s=1,3,5,\dots} (-)^{(s-1)/2}$$

$$\int\limits_{-\infty}^{\infty} \left[\sum_{n=0}^{\infty} \hat{f}_n(k;\omega)\, H_n(k) \right] Q_{k-1/2}^{(1)}(\cos\Theta)e^{-\mathrm{i}(s-1)k\pi}\, k\,\mathrm{d}k$$

$$+ \frac{1}{2\pi} \sum_{s=2,4,\ldots} (-)^{s/2} \int\limits_{-\infty}^{\infty} \left[\sum_{n=0}^{\infty} \hat{f}_n(k;\omega)\, H_n(k) \right] Q_{k-1/2}^{(2)}(\cos\Theta)e^{-\mathrm{i}\,sk\pi}\, k\,\mathrm{d}k. \quad \text{(A.11)}$$

The integrands in the terms of these series can be identified as wave constituents traveling along the surface or boundary, the representations of which can be obtained by techniques from semi-classical analysis. Indeed, s is referred to as the (multi-orbit) arrival number while we distinguish the orientation of propagation in the two series. The term $s = 1$ corresponds with waves that propagate from source to receiver along the minor arc; the term $s = 2$ corresponds with waves that propagate from source to receiver along the major arc. At $\Theta = 0$ and $\Theta = \pi$ the traveling wave Legendre functions have logarithmic singularities, namely $\log\Theta$ at $\Theta = 0$ and $\log(\pi - \Theta)$ at $\Theta = \pi$; the singularities cancel when taking the sums together.

A.2.2. Debye expansion

Here, we describe how to relate the sum of eigenfunctions in (A.11) to a kernel closely related to the propagator. First, we note that

$$\int\limits_0^{\infty} {}_n f_l(t)e^{-\mathrm{i}\omega t}\,\mathrm{d}t = \frac{1}{\omega_n(k)^2 - \omega^2}$$

(cf. (2.8)). We substitute, again, $k = \omega p$, and introduce the Debye expansion for p fixed. Thus we can isolate the different regimes:

- **Diving** $(R/c(R) < p < 1/c(1))$: The dispersion relations satisfy,

$$\omega_n(p) := \omega_n = \left(n + \frac{5}{4} \right) \pi\overline{\tau}(p)^{-1}, \quad \overline{\tau}(p) = \int\limits_{R^\star}^1 \beta(r';p)\,\mathrm{d}r'$$

(cf. (A.4)) and the WKB eigenfunctions yield

$$U_n(r;p)U_n(r_0;p) = 4T^{-1}(rr_0)^{-1}$$
$$c(r)^{-1}\rho(r)^{-1/2}c(r_0)^{-1}\rho(r_0)^{-1/2}\beta^{-1/2}(r;p)\beta^{-1/2}(r_0;p)$$
$$\sin[(n+\tfrac{5}{4})\pi\widetilde{\tau}(r;p)/\overline{\tau}(p)]\sin[(n+\tfrac{5}{4})\pi\widetilde{\tau}(r_0;p)/\overline{\tau}(p)],$$

where

$$\omega\widetilde{\tau}(r;p) = \omega \int\limits_{R^\star}^r \beta(r';p)\,\mathrm{d}r' + \frac{\pi}{4},$$

to leading order. In fact, $2\overline{\tau}\,T^{-1}$ cancels against $(1 - c_n^{-1}C_n)^{-1}$; hence, we premultiply $U_n(r;p)U_n(r_0;p)$ by $\overline{\tau}$ before analyzing the summation. The summation,

$$(rr_0)c(r)\rho(r)^{1/2}c(r_0)\rho(r_0)^{1/2}\beta^{1/2}(r;p)\beta^{1/2}(r_0;p)\frac{T}{4\overline{\tau}} \sum_{n=0}^{\infty} \frac{U_n(r;p)U_n(r_0;p)}{\omega_n(p)^2 - \omega^2}$$

can be carried out to yield (see [18, p. 515])

$$\frac{1}{\overline{\tau}} \sum_{n=0}^{\infty} \frac{\sin[(n+\frac{5}{4})\pi \widetilde{\tau}(r;.)/\overline{\tau}] \sin[(n+\frac{5}{4})\pi \widetilde{\tau}(r_0;.)/\overline{\tau}]}{[(n+\frac{5}{4})\pi/\overline{\tau}]^2 - \omega^2}$$
$$= \frac{1}{8\,\mathrm{i}\,\omega} \sum_{i=1}^{\infty} \exp\left[-\mathrm{i}\,\omega \tau_i(r,r_0;p) + \mathrm{i}\,N_j \frac{\pi}{2}\right], \tag{A.12}$$

where

$$\tau_1(r,r_0;p) = \left|\int_{r_0}^{r} \beta(r';p)\,\mathrm{d}r'\right|$$

$$\tau_2(r,r_0;p) = \int_{R^\star}^{r_0} \beta(r';p)\,\mathrm{d}r' + \int_{R^\star}^{r} \beta(r';p)\,\mathrm{d}r',$$

$$\tau_3(r,r_0;p) = \int_{r_0}^{1} \beta(r';p)\,\mathrm{d}r' + \int_{r}^{1} \beta(r';p)\,\mathrm{d}r',$$

$$\tau_4(r,r_0;p) = 2\int_{R^\star}^{1} \beta(r';p)\,\mathrm{d}r' - \left|\int_{r_0}^{r} \beta(r';p)\,\mathrm{d}r'\right|,$$

$$\tau_i(r,r_0;p) = \tau_{i-4}(r,r_0;p) + 2\int_{R^\star}^{1} \beta(r';p)\,\mathrm{d}r', \quad i = 5,6,\ldots$$

while

$$N_1 = 0, \qquad N_2 = 1, \qquad N_3 = 0, \qquad N_4 = 1, \qquad N_i = N_{i-4} + 1, \quad i = 5,6,\ldots$$

- **Reflecting** $(0 < p < R/c(R))$: The dispersion relations satisfy,

$$\omega_n(p) := \omega_n = (n+1)\pi\overline{\tau}(p)^{-1}, \quad \overline{\tau}(p) = \int_{R}^{1} \beta(r';p)\,\mathrm{d}r'$$

(cf. (A.6)) and an analogous computation as (A.12) yields the same result in the reflecting regime except that $N_i = 0$, $i = 1,2,\ldots$ in this case.
- **Grazing**: To leading order, using the asymptotics of the Airy functions, we have

$$V_n(r;p) = 2B_1\beta^{-1/2}\sin\left(\omega_n(p)\int_{R^\star}^{r} \beta\,\mathrm{d}r' + \pi/4\right)$$

$$+ 2B_2\beta^{-1/2}\sin\left(\omega_n(p)\int_{R^\star}^{r} \beta\,\mathrm{d}r' - \pi/4\right)$$

$$= 2B_1\beta^{-1/2}\sin((n+\delta)\pi\widetilde{\tau}_+(r)/\overline{\tau}) + 2B_2\beta^{-1/2}\sin((n+\delta)\pi\widetilde{\tau}_-(r)/\overline{\tau}),$$

where $\delta = 13/12$ (while in the diving case it was $5/4$) and

$$\omega \widetilde{\tau}_{\pm}(r; p) = \omega \int_{R^{\star}}^{r} \beta(r'; p) \, dr' \pm \frac{\pi}{4}.$$

We then apply the argument used to obtain (A.12) with $5/4$ replaced by δ and replacing each $\widetilde{\tau}$ by the appropriate $\widetilde{\tau}_{\pm}$. In the end, we get the same result except with a different index N_i depending on δ. To unify notation, we can introduce the appropriate N_i, which will not be an integer in this case. To combine the four terms in the end, one uses $B_1^2 = 3/(4T)$, $B_2^2 = 1/4T$, and $\sqrt{3} = \exp[i\pi/6] + \exp[-i\pi/6]$. More details showing that the leading order contribution of a periodic grazing ray has a similar form as the other cases is in Appendix B.

We apply to (A.11) the Debye expansion described above to obtain a form more closely resembling a wave propagator:

$$\sum_{n=0}^{\infty} \hat{f}_n(k; \omega) H_n(k) = k^{-2} (rr_0 c(r) c_0(r))^{-1} (\rho(r)\rho(r_0)\beta(r; p)\beta(r_0; p))^{-1/2} \frac{1 - c_n^{-1} C_n}{2 \, i \, \omega}$$

$$\sum_{i=1}^{\infty} \exp\left[-i\omega\tau_i(r, r_0; p) + i\, N_i \frac{\pi}{2} \right].$$

Next, we change variables of integration from k to p. We encounter the Jacobian (cf. (A.7))

$$\frac{dp}{dk} = \frac{1}{c_n} \left(\frac{C_n}{c_n} - 1 \right) \frac{1}{k},$$

so that

$$p^{-1} \, dp = (1 - c_n^{-1} C_n) \, k^{-1} \, dk.$$

The path of integration is beneath the real axis, while taking $\omega > 0$. After making the substitution into (A.11), and inserting the leading order expansion (valid for Re $p > 0$),

$$Q_{\omega p - 1/2}^{(1)}(\cos \Theta) \simeq \left(\frac{1}{2\pi\omega p \sin \Theta} \right)^{1/2} e^{-i(\omega p \Theta - \pi/4)}$$

(cf. (A.9)) to obtain (cf. (A.11)) a single term in the sum

$$\frac{1}{4\pi} \, (-)^{(s-1)/2} (rr_0 c(r) c_0(r))^{-1} (\rho(r)\rho(r_0))^{-1/2}$$

$$\int_{-\infty}^{\infty} (\beta(r; p)\beta(r_0; p))^{-1/2} \left[\sum_{i=0}^{\infty} \exp\left[-i\omega\tau_i(r, r_0; p) + i\, N_i \frac{\pi}{2} \right] \right]$$

$$Q_{\omega p - 1/2}^{(1)}(\cos \Theta) e^{-i\omega(s-1)p\pi} \, p^{-1} \, dp$$

$$\simeq \frac{1}{4\pi} (-)^{(s-1)/2} (rr_0 c(r) c(r_0))^{-1} (2\pi\rho(r)\rho(r_0) \sin \Theta)^{-1/2}$$

$$\int (\beta(r; p)\beta(r_0; p))^{-1/2} \sum_{i=1}^{\infty} \exp[-i\omega(\tau_i(r, r_0; p) + p\Theta + (s-1)p\pi)]$$

$$\exp[i(\pi/4)(2N_i - 1)](\omega p)^{-3/2} \, dp. \quad (A.13)$$

Appendix B. Periodic grazing ray

In this appendix, we will provide a more detailed analysis on the contribution of a periodic grazing ray to the trace formula. We show that the leading order term (as $\omega \to \infty$) has the same form as in Proposition 2.3. Our analysis closely follows [6, Chapter 1]. The computations involve Airy functions, which is natural in light of the Airy-type parametrices constructed in [2,24,45] microlocally near a grazing ray. We cannot use those parametrices since they are only local and we need a global parametrix to take the trace.

We assume U satisfies the inner boundary condition and U_n satisfies both boundary conditions. We will need to compute

$$D = D(p, \omega) := U_n T|_{r=1} - U_n T|_{r=R} = U_n T|_{r=1},$$

and then replace ω_n by a general ω. Using the asymptotic computation to sum the eigenfunctions computed in Appendix A or using the computation in [48], we have the representation

$$\hat{G}(x, x_0, \omega) = \frac{1}{2\pi} \sum_{l=0}^{\infty} \frac{l + \frac{1}{2}}{l(l+1)} D^{-1} \mathbf{D}_l (\mathbf{D}_l)_0 P_l(\cos \Theta).$$

Let A_r and B_r denote two linearly independent solutions to leading order for the equation (2.3). We will later pick using the Airy function A_+

$$A_r = A_r(\omega, p) = 2\pi^{1/2} \mu^{-1/2} r^{-1} \chi^{1/6} (-\beta^2)^{-1/4} \mathrm{A}_+ (\omega^{2/3} \chi^{2/3}(r)),$$

$$\chi(r) = -(3/2) \int_{R^\star}^{r} (-\beta^2)^{1/2} \, \mathrm{d}r,$$

and similarly for B_r but using the Airy function A_-. We can set

$$U_n = C_1 A_r + C_2 B_r$$

for constants C_1 and C_2 that depend on p and ω. The Neumann inner boundary condition to leading order is

$$\partial_r U_n|_{r=R} = 0$$

and one possible solution is

$$U_n = B_R' A_r - A_R' B_r$$

where a specific eigenvalue ω_n is being used. Thus, we get

$$\frac{1}{\mu} T = \partial_r U = B_R' A_r' - A_R' B_r'. \tag{B.1}$$

Since U_n is an eigenfunction, then $\partial_r U_n = 0$ at $r = 1$ gives

$$\frac{B_R'}{A_R'} = \frac{B_1'}{A_1'}$$

when $\omega = \omega_n$.

Thus, we can write

$$U_n(r) = A'_R \left(\frac{B'_1}{A'_1} A_r - B_r \right)$$
$$= \frac{A'_R}{A'_1} \left(B'_1 A_r - A'_1 B_r \right),$$

and we get

$$U_n(1) = \frac{A'_R}{A'_1} W(A, B),$$

where $W(A, B)$ is the Wronskian of A_r, B_r and is independent of r. We can now compute using (B.1)

$$\mu^{-1} D(\omega) = U_n(1) T(1) = \frac{A'_R}{A'_1} W(A, B) B'_R A'_1 \left(1 - \frac{A'_R}{A'_1} \frac{B'_1}{B'_R} \right)$$
$$= A'_R B'_R W(A, B) \left(1 - \frac{A'_R}{A'_1} \frac{B'_1}{B'_R} \right).$$

Thus,

$$\frac{\mu_1 U(r) U(r_0)}{D} = \frac{1}{W(A, B)} \left(\frac{B'_R}{A'_R} A_r - B_r \right) \left(A_{r_0} - \frac{A'_R}{B'_R} B_{r_0} \right) \sum_k \left(\frac{A'_R}{B'_R} \frac{B'_1}{A'_1} \right)^k$$

Note that even though $B'_s = \frac{d}{dr}\big|_{r=s} B_r$ and similarly for A'_s, we have

$$\frac{B'_s}{A'_s} = \frac{A_-(\omega^{2/3} \chi^{2/3}(s))}{A_+(\omega^{2/3} \chi^{2/3}(s))}.$$

When computing the trace $\int_R^1 D^{-1} U(r) U(r) \rho dr$, we need to compute the quantities

$$l_{-1} = \int_R^1 A_r^2 \rho r^2 \, dr, \qquad l_0 = \int_R^1 A_r B_r \rho r^2 \, dr, \qquad l_1 = \int_R^1 B_r^2 \rho r^2 \, dr$$

to leading order as $\omega \to \infty$. If these quantities are a symbol in ω, as well as B'_R/A'_R for p near the grazing ray value $R/c(R)$, then we can just apply stationary phase to $(B'_1/A'_1)^k$ using the asymptotic expansion of the Airy function at ∞ by treating the rest of the integrand as the amplitude in the stationary phase calculation. The quantities l_i are order 0 and can be computed to leading order as done in (A.3), and the principal term is related to the travel time.

Via the analogous computation when taking the trace as in the proof of Proposition 2.3 using the Poisson summation formula for the sum over l, we obtain to leading order as $\omega \to \infty$

$$\int \hat{G}(x, x, \omega) \, dx \simeq \sum_{j=-1}^{1} \sum_i \sum_s V_{is}^{(j)}(\omega)$$

where

$$V_{is}^{(j)} = \int e^{i \pi \omega p s} a_s^{(j)}(p, \omega) \left(\frac{B'_1}{A'_1} \right)^i \left(\frac{A'_R}{B'_R} \right)^{i+j} \, dp$$

and

$$a_s^{(j)}(p,\omega) = \frac{1}{2\pi W(A,B)}(-)^{(s-1)/2}\omega^2 p^{1/2} l_j$$

is a symbol of order two. This form of the solution operator can be compared to [6, (1.14) and (3.2)]. Let us rewrite this as

$$V_{is}^{(j)} = \omega^2 \int b_{ijs}(p)\left(\frac{A_R'(p)}{B_R'(p)}\right)^{i+j}\,\mathrm{d}p = \omega^2 \int \left(\frac{d}{dp}\int_{-\infty}^p b_{ijs}(y)\,\mathrm{d}y\right)\left(\frac{A_R'(p)}{B_R'(p)}\right)^{i+j}\,\mathrm{d}p$$

with $b_{ijs}(p)$ defined in the obvious way in comparison to the previous formula for $V_{is}^{(j)}$. We integrate by parts to obtain

$$= \omega^2 \left[\int_{-\infty}^p b_{ijs}(y)\,\mathrm{d}y\left(\frac{A_R'(p)}{B_R'(p)}\right)^{i+j}\right]_{p=-\infty}^\infty$$

$$-\omega^2 \int \mathrm{d}y \int_{-\infty}^p b_{ijs}(y)(i+j)\left(\frac{A_R'(p)}{B_R'(p)}\right)^{i+j-1}\frac{B_R'\frac{d}{dp}A_R' - A_R'\frac{d}{dp}B_R'}{(B_R')^2}\,\mathrm{d}p.$$

The first term is

$$\omega^2 \int_{-\infty}^\infty b_{ijs}(y)\,\mathrm{d}y\left(\frac{A_R'(\infty)}{B_R'(\infty)}\right)^{i+j} = \omega^2 \int_{-\infty}^\infty b_{ijs}(y)\,\mathrm{d}y$$

since $A_+'(\infty)/A_-'(\infty) = 1$. This is the main "classical" term where we can apply the method of steepest descent as done in the proof of Proposition 2.3 and using the asymptotics of $A_\pm$ as in [6, Chapter 1]. We just need to verify that the other term is the sum of an analogous "classical" term and a term that is lower order, but where stationary phase cannot be applied near the grazing ray.

After using the Airy equation, the second term becomes

$$\omega^2 \int \mathrm{d}y \int_{-\infty}^p b_{ijs}(y)(i+j)\frac{(A_R'(p))^{i+j-1}}{(B_R'(p))^{i+j+1}}W(A,B)(d_p\chi_R^{2/3})\chi_R^{2/3}\omega^{4/3}\,\mathrm{d}p$$

$$= \omega^{10/3}(i+j)W(A,B)\int \tilde{b}_{ijs}(p,\omega)\frac{(A_R'(p))^{i+j-1}}{(B_R'(p))^{i+j+1}}(d_p\chi_R^{2/3})\chi_R^{2/3}\,\mathrm{d}p,$$

where the subscript R on χ_R means its evaluated at $r = R$ and

$$\tilde{b}_{ijs}(p,\omega) = \int_{-\infty}^p b_{ijs}(y)\,\mathrm{d}y$$

Our integrand contains terms of the form

$$A_\pm'(\omega^{2/3}\chi_R^{2/3}(p))$$

so we use the substitution

$$q = \chi_R^{2/3}(p), \qquad \mathrm{d}q = d_p \chi_R^{2/3}(p)\,\mathrm{d}p$$

so $p = p(q)$ is a function of q and we get

$$= \omega^{10/3}(i+j)W(A,B)\int \tilde{b}_{ijs}(q,\omega)\frac{(\mathrm{A}'_+(\omega^{2/3}q))^{i+j-1}}{(\mathrm{A}'_-(\omega^{2/3}q))^{i+j+1}}q\,\mathrm{d}q.$$

Now we substitute

$$w = \omega^{2/3}q$$

to obtain

$$= \omega^2(i+j)W(A,B)\int \tilde{b}_{ijs}(\omega^{-2/3}w,\omega)\frac{(\mathrm{A}'_+(w))^{i+j-1}}{(\mathrm{A}'_-(w))^{i+j+1}}w\,\mathrm{d}w.$$

Near the p value $p_g := R/c(R)$ corresponding to a periodic grazing ray is where stationary phase fails. If $p = p_G$, then $q = 0$. Thus, we will do a Taylor series about $w = 0$ and we have

$$\tilde{b}_{ijs}(\omega^{-2/3}w,\omega) = \tilde{b}_{ijs}(0,\omega) + \omega^{-2/3}\tilde{c}_{ijs}(\omega^{-2/3}w,\omega)$$

Applying the proof of [6, Proposition 9], the second term is indeed of order $\omega^{-2/3}$ and lower order than the principal term and can be disregarded. In fact, one can continue the Taylor expansion of the second terms and actually obtain lower order terms in the trace formula but we do not pursue this. Thus, taking the principal term gives us

$$\simeq \omega^2(i+j)W(A,B)\int \tilde{b}_{ijs}(0,\omega)\frac{(\mathrm{A}'_+(w))^{i+j-1}}{(\mathrm{A}'_-(w))^{i+j+1}}\,\mathrm{d}w.$$

Using an analogous computation as in the proof of [6, Proposition 9], we have

$$(i+j)\int\limits_{-\infty}^{\infty} W(A,B)\frac{(\mathrm{A}'_+(w))^{i+j-1}}{(\mathrm{A}'_-(w))^{i+j+1}}w\,\mathrm{d}w = \left(\frac{A'_R(\infty)}{B'_R(\infty)}\right)^{i+j} = 1.$$

We are then left with

$$V_{ik}^j \simeq \omega^2\int\limits_{-\infty}^{\infty} b_{ijs}(y)\,\mathrm{d}y - \omega^2\tilde{b}_{ijs}(q=0,\omega) = \omega^2\int\limits_{p_g}^{\infty} b_{ijs}(y)\,\mathrm{d}y,$$

where our stationary phase computation can be applied. It is interesting to note that near the grazing ray, the lower order terms involve quantities of the form

$$\int w^m \frac{(\mathrm{A}'_+(w))^k}{(\mathrm{A}'_-(w))^k}\,dw$$

as also discussed in [5].

Appendix C. Class of metrics satisfying (A1)-(A4)

Here, we provide a class of metrics satisfying all the assumptions above. We consider the example given in [11, Section 3.7]. Pick c such that

$$(c/r)^{-2} = a + b\ln(r),$$

for parameters $a, b > 0$ so (A4) is satisfied. Then

$$\alpha(p) = \int_{R^*}^{1} \frac{p}{r'\sqrt{(c/r')^{-2} - p^2}}\, dr' = \int_{R^*}^{1} \frac{p}{r'\sqrt{a + b\ln(r) - p^2}}$$

where R^* depends on p. We may then compute

$$\alpha(p) = \begin{cases} 2p[\sqrt{a - p^2} - \sqrt{a + b\ln(R) - p^2}] & \text{if } 0 < p < \sqrt{a + b\ln(R)} \\ 2p\sqrt{a - p^2} & \text{if } \sqrt{a + b\ln(R)} < p < \sqrt{a} \end{cases}$$

since in the reflecting ray case, $R^* = R$ and in the diving ray case (see Appendix A), R^* is determined by the condition $(c(R^*)/R^*)^{-2} - p^2 = 0$. In that case,

$$\alpha'(p) = 2\sqrt{a - p^2} - 2p^2/\sqrt{a - p^2}$$

which vanishes only if $p^2 = a/2$. Similarly, one may verify that $\alpha'(p)$ does not vanish in the reflecting region except for countable many values so (A3) is satisfied.

To get a periodic ray, we need $\alpha = 2\pi q$ for some $q \in \mathbb{Q}$. This would imply

$$p^2(a - p^2) = \pi^2 q^2.$$

Solving for p, we obtain

$$p^2 = \frac{a \pm \sqrt{a^2 - 4\pi^2 q^2}}{2}$$

Hence, the periodic diving rays are in one-to-one correspondence with elements of $\mathbb{Q}$ such that

$$a^2 - 4\pi^2 q^2 \geq 0.$$

If we pick a such that $a \notin \pi\mathbb{Q}$, then $p^2 \neq a/2$ for a periodic ray and the clean intersection hypothesis is satisfied. A similar calculation gives the result for reflecting rays.

Remark C.1. Note that the proof of the main theorems does not require recovery of all the lengths in $\mathrm{lsp}(c)$. Hence, even if the clean intersection hypothesis fails on a finite number of rays and one does not recover those lengths with the trace formula, the theorems are still valid with the same proof.

For the geometric spreading injectivity condition, consider $f(p) = \alpha'(p)/p^2$ and let $\tilde{a} = a + b\ln(R) < a$ if $b > 0$. A quick calculation shows $f'(p) < 0$ in the diving regime $(p > \sqrt{\tilde{a}})$ so $\alpha'(p)/p^2$ is injective for $p > \sqrt{\tilde{a}}$. For $0 < p < \sqrt{\tilde{a}}$, a tedious computation shows that

$$f(p) > \lim_{p \to \sqrt{\tilde{a}}^+} f(p).$$

Since f is decreasing for $p > \sqrt{\tilde{a}}$, this implies that if $f(p_1) \neq f(p_2)$ if $p_1 \in (0, \sqrt{\tilde{a}})$ and $p_2 \in (\sqrt{\tilde{a}}, \sqrt{a})$. Hence, the geometric spreading injectivity condition (A2) is satisfied.

Appendix D. General framework for symmetries in a manifold

In this appendix, we consider more general situations for trace formulas when the manifold has symmetries given by a compact Lie group and we show how the quantities appearing in Proposition 2.3 are special cases of a general framework. Our constructions here are inspired by the work in [14,15,7,10,27].

Let (M, g) be a compact Riemannian manifold with boundary satisfying the same assumptions as in [30]. In fact, all that is necessary is that one may construct a parametrix in the same form as [30] for the Neumann wave propagator. We assume that a compact Lie group G has a symplectic group action on T^*M and it accounts for all the symmetries of the Hamiltonian $p(x, \xi) = |\xi|_g^2$, which is the principal symbol of Δ_g. That is, $p(g.z) = p(z)$ for each $z \in T^*M$ and $g \in G$, where $g.z$ or gz denotes the group action. The assumption is merely that G accounts for all such symmetries.

The main assumption we make is that the connected components of C_T are the G-orbit of a particular periodic bicharacteristic γ and it has no other symmetry; i.e. assuming C_T is already connected, $C_T = G\gamma := \{g.x; g \in G$ and $x \in \mathrm{Im}(\gamma)\}$. We note that this assumption captures generic situations since on general manifolds without symmetry, C_T has dimension 1 (see [23, p. 61]), and an increase in dimension should only come from a group symmetry.

Next, we set $L = I - \mathrm{d}\Phi^T(m)$ for $m \in C_T$. In the context of group symmetries, the map $I - \mathrm{d}\Phi^T : T_m S^*M/\ker(L) \to T_m S^*M/\ker(L)$ is in general not an isomorphism anymore so [23, lemma 4.4] does not apply. However, the induced map

$$I - (\mathrm{d}\Phi^T)^\perp : T_m T^*M/\ker(L^2) \to T_m T^*M/\ker(L^2)$$

will generically be an isomorphism [7, appendix A.1], [14,15,10]. The above map is the linear Poincaré map appearing in [23,30] but is technically not the same map appearing in our theorems since in our case, $\ker(L^2) = T_m T^*M$. We also assume the *clean intersection hypothesis* described in [23,30] so that C_T is a submanifold and $\ker(L) = T_m C_T$. See Remark 4.3 for a geometric description of the clean intersection hypothesis in spherical symmetry.

The new part of the trace formula will come from $\ker(L^2)/\ker(L)$ whose presence is already seen in the calculation of the normal Hessian for the stationary phase analysis in section 3.1. Based on the arguments in [14,15], there is a geometric scalar quantity defined on C_T, that is nonvanishing and associated with the action of $I - \mathrm{d}\Phi^T$ on $\ker(L^2)/\ker(L)$ [14, appendix]. We denote this action by

$$I - P_{[\gamma]} : \ker(L^2)/\ker(L) \to \mathrm{Im}(I - P_{[\gamma]}) \subset \ker(L),$$

which turns out to be an isomorphism and is the quantity appearing in section 2 in our spherically symmetric setting. This is a quantity directly related to periodic orbits in the reduced phase space formally written as T^*M/G and defined in [1, Chapter 4], and $|I - P_{[\gamma]}|$ will stay constant over $C_T = [\gamma]$. For our $SO(3)$ action and the geometric assumptions described in section 2, $\ker(L^2)/\ker(L)$ is two-dimensional with one of the elements being trivial, determined by the infinitesimal generator of dilations in the dual variables (see [23, p. 70] for more details). The other basis element corresponds to the Jacobi field that perturbs the angular momentum (ray parameter p).

In summary, our trace formula fits into a general Lie group framework and may be compared to the formula appearing in [15] albeit in a different setting.

Appendix E. A remark about spherical symmetry

Any spherically symmetric manifold is in fact of the form we consider – radially conformally Euclidean.

Proposition E.1. *Let $A \subset \mathbb{R}^n$ be an annulus (difference of two cocentric balls) and $g \in C^k$ a rotationally symmetric metric in the sense that $g = U^*g$ for all $U \in SO(n)$. Then (A, g) is isometric to a Euclidean annulus with the Euclidean metric multiplied with a conformal factor $c^{-2}(|x|)$ with $c \in C^k$.*

Proof. In this proof it is more convenient to write $c^{-2} = \eta$. This notation is not used elsewhere.

By rotational symmetry it suffices to consider the metric at points $z_r = (r, 0, \ldots, 0)$ for $r \in [R, 1]$. Near z_r we take local coordinates $(x, y) \in \mathbb{R} \times \mathbb{R}^{n-1}$ such that (x, y) represents the point $(r + x, y)$ on A. In these coordinates we can write the metric at z_r as the matrix

$$g_{z_r} = \begin{pmatrix} a(r) & b(r)^T \\ b(r) & C(r) \end{pmatrix},$$

where $a(r)$, $b(r)$ and $C(r)$ are a number, a vector and a matrix depending on r. We will drop the argument (r) where it is implicitly clear. We will first show that b is identically zero (if $n \geq 3$) or becomes zero after applying a diffeomorphism that preserves rotational symmetry ($n = 2$) and $C(r) = c(r)I$ for some scalar function c (trivial for $n = 2$).

We first consider the case $n \geq 3$. For any $R \in SO(n-1)$ we have $U_R := 1 \oplus R \in SO(n)$. Since g_{z_r} must be invariant under U_R, we have $Rb = b$ and $RCR^{-1} = C$. But this holds for all $R \in SO(n-1)$, so $b = 0$ and C is a multiple of identity.

We then turn to the case $n = 2$. Now b and C are scalars, and we write $C = c$. By positive definiteness of the metric we have $a > 0$, $c > 0$, and $b^2 < ac$.

We write points on A in polar coordinates (r, θ) and define a function $F_\varphi \colon A \to A$ parametrized by a function $\varphi \colon [R, 1] \to \mathbb{R}$ by setting $F(r, \theta) = (r, \theta + \varphi(r))$. If φ is C^k, then F is clearly a C^k diffeomorphism. After we fix $r_0 \in [R, 1]$, we may assume that $\varphi(r_0) = 0$ by rotational symmetry, so that $F(z_{r_0}) = z_{r_0}$. In the Euclidean coordinates (x, y) near z_{r_0} we have

$$DF_{z_{r_0}} = \begin{pmatrix} 1 & 0 \\ \alpha(r_0) & 1 \end{pmatrix}$$

where $\alpha(r) = r\varphi'(r)$, which implies

$$F_\varphi^* g_{z_{r_0}} = \begin{pmatrix} a + 2b\alpha + c\alpha^2 & b + c\alpha \\ b + c\alpha & c \end{pmatrix}. \tag{E.1}$$

If we choose the function φ so that $r\varphi'(r) = -b(r)/c(r)$, the metric (E.1) becomes diagonal. Since c is bounded from below uniformly on $[R, 1]$, the function $\varphi(r) = -\int^r b(s)/sc(s)\mathrm{d}s$ is well defined and C^{k+1} if the original metric is C^k; additive constants are irrelevant, since they correspond to rotations of the entire annulus.

We have now shown that the metric can be assumed to have the form

$$g_{z_r} = \begin{pmatrix} a(r) & 0 \\ 0 & c(r)I \end{pmatrix}.$$

If $\rho \colon [R, 1] \to \mathbb{R}$ is a strictly increasing C^1 function, we define the change of variable (again in polar coordinates) $G_\rho(r, \theta) = (\rho(r), \theta)$. The function ρ is a diffeomorphism and we denote $\sigma = \rho^{-1}$. We will later choose ρ so that $\rho(1) = 1$ and $\rho(R) > 0$, which makes $G_\rho \colon A \to \bar{B}(0, 1) \setminus B(0, \rho(R))$ a diffeomorphism.

A simple calculation shows that

$$(G_\rho^{-1})^* g_{z_r} = \begin{pmatrix} a(\sigma(r))\sigma'(r)^2 & 0 \\ 0 & c(\sigma(r))(\sigma(r)/r)^2 I \end{pmatrix}. \tag{E.2}$$

If we construct ρ so that

$$a(\sigma(r))\sigma'(r)^2 = c(\sigma(r))(\sigma(r)/r)^2, \tag{E.3}$$

the metric (E.2) is a multiple of the identity matrix (conformally Euclidean) and

$$(\bar{B}(0,1) \setminus B(0,\tau), (G_\rho^{-1})^* g)$$

is a manifold of the desired form. We will see that $\rho \in C^{k+1}$, and this shows the regularity claim.

For convenience, we change variable from r to $s = \sigma(r)$. Condition (E.3) now becomes

$$\frac{\mathrm{d}}{\mathrm{d}s} \log \rho(s) = \frac{1}{s} \sqrt{\frac{a(s)}{c(s)}}.$$

We thus choose

$$\rho(s) = \exp\left(\int_1^s \frac{1}{t} \sqrt{\frac{a(t)}{c(t)}} \mathrm{d}t\right). \tag{E.4}$$

Since the integrand is strictly positive and C^k, the function ρ is a strictly increasing C^{k+1} function as claimed. We also claimed earlier that ρ satisfies $\rho(1) = 1$ and $\rho(R) > 0$, and these properties can be read in the representation (E.4). $\square$

Remark E.2. The diffeomorphism of Proposition E.1 is in fact radial if $n \geq 3$. It will also be necessarily radial in two dimensions if the metric is invariant under the action of $SO(2)$, but also that of $O(2)$.

Appendix F. Some exotic spherically symmetric geometries

Our results assumed several geometric hypotheses. In this appendix we explore some problematic spherically symmetric geometries which are ruled out by our assumptions.

First, recall that countability of the length spectrum was shown under the Herglotz and countable conjugacy conditions. It is not known whether the length spectrum can be uncountable without these assumptions.

Let us then see concrete examples where our assumptions are violated and it leads to problematic behavior:

Example F.1. If the derivative $\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{r}{c(r)}\right)$ vanishes in an open set of radii, then that part of the manifold is isometric to a cylinder $S^2(0,a) \times (0,b)$ for some $a, b > 0$. The great circles of S^2 with the second variable constant are periodic geodesics, and they all have the same length $T = 2\pi a$. The dimension of the corresponding fixed point set C_T is 3. This manifold is trapping and the periodic orbits in question do not reach the boundary. The Herglotz condition is violated.

Example F.2. Consider the closed hemisphere of S^3 or any other S^n, $n \geq 2$. Now all broken rays are periodic, and all primitive periods are 2π. The primitive length spectrum is degenerated into a single point. The Herglotz condition fails at the boundary (but only there), and the endpoints of *all* maximal geodesics

are conjugate. The fixed point set has full dimension; for $n = 3$ we have $\dim(C_{2\pi}) = 5$. This manifold is non-trapping.

Example F.3. Consider the closed hemisphere $H \subset S^2$. In polar coordinates we can identify H near the boundary (equator) with $(R, 1] \times [0, 2\pi]$, where the angles 0 and 2π are identified in the obvious way. For any $L \in (0, 2\pi)$, we can take the submanifold $(R, 1] \times [0, L]$ and identify angles 0 and L. If $L \notin \pi\mathbb{Q}$, then no broken ray near the boundary is periodic. By angular rescaling we may write this as a rotation invariant metric near the boundary of the closed disc $\bar{D}$ and continue it to a rotation invariant metric in the whole disc. Now there is an open set of points on the sphere bundle containing no periodic orbits, and this open set can be made large. Only rays going close enough to the center of the disc will contribute to the length spectrum. This manifold is also non-trapping, the Herglotz condition fails at the boundary, and the countable conjugacy condition is violated.

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