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## ABSTRACT

We analyze an inverse problem associated with the time-harmonic Rayleigh system on a flat elastic half-space concerning the recovery of Lamé parameters in a slab beneath a traction-free surface. We employ the Markushevich substitution, while the data are captured in a Jost function, and we point out parallels with a corresponding problem for the Schrödinger equation. The Jost function can be identified with spectral data. We derive a Gel'fand-Levitan type equation and obtain uniqueness with two distinct frequencies.



## I. INTRODUCTION

In this paper, we study an inverse problem associated with the Rayleigh system on a flat elastic half-space concerning the conditional unique recovery of the Lamé parameters in a slab beneath a traction-free surface. In the process, we point out parallels with a corresponding problem for the Schrödinger equation. The analysis involves the study of spectral properties of a matrix Sturm-Liouville operator on the half-line with a Robin-type boundary condition associated with elastic surface waves of Rayleigh type. This operator is non-self-adjoint and contains the spectral parameter in the boundary condition. It originates from the standard Rayleigh boundary value problem by means of the so-called Markushevich substitution. In this boundary value problem, the Lamé parameters, $\lambda$ and $\mu$, and the density of mass, $\rho$, appear as material parameters and are assumed to be functions of the boundary normal coordinate or "depth," $Z$, say. In our analysis, however, $\rho$ will not play a role. In previous work,[1] we analyzed the inverse problem associated with the Rayleigh system using the semiclassical spectrum as the data. Essentially, this semiclassical inverse problem allows for the recovery of one of the Lamé parameters, namely, the shear modulus $\mu$.

Using also an adjoint Markushevich substitution, we develop a spectral theory, which inherits many features of self-adjoint matrix-valued Sturm-Liouville problems, as in Ref. 2. This follows from the fact that the original Rayleigh operator is self-adjoint. We construct a Gel'fand-Levitan equation. As an aside, the approach presented here allows for a generalization from a traction-free surface to an isotropic solid-fluid boundary, leading to Scholte-Gogoladze waves, assuming that the fluid is homogeneous and known. The Markushevich substitution was introduced by Markushevich[3–5] following the ideas of Pekeris[6] and was recently revisited in Ref. 7.

The idea of applying the Markushevich substitution originates in the work of Beals, Henkin, and Novikova,[8] where a spectral analysis was performed in the context of exponentially increasing quantities, $\hat{\lambda}(Z) = \rho(Z)^{-1}\lambda(Z), \hat{\mu}(Z) = \rho(Z)^{-1}\mu(Z)$ as $Z \to -\infty$, with $Z \in (-\infty, 0]$, which differs considerably from the assumptions in Ref. 9 where $\hat{\lambda}(Z)$, $\hat{\mu}(Z)$ are constant, with values $\hat{\lambda}_0$, $\hat{\mu}_0$, respectively, beneath a certain depth $Z = -H$ while *preserving* the Rayleigh system as $Z \to -\infty$ and enabling application in seismology with the slab signifying Earth's crust. Our data are necessarily different from the data considered by Beals, Henkin, and Novikova. We consider the recovery of $\hat{\lambda}$ and $\hat{\mu}$ in a *slab* with known thickness, $H$, from the Jost function. We show that the Gel'fand-Levitan equation[10] can be constructed in our case and has a unique solution.

In the inverse problem, boundary spectral data directly encode the Weyl matrix of the transformed problem and, equivalently, the Neumann-to-Dirichlet (ND) map associated with the original Rayleigh system. Boundary spectral data are, however, insufficient to guarantee unique recovery of both Lamé parameters in the slab. In fact, we require the Jost function as the *spectral data*. We let $x$ denote the coordinate tangential to the boundary with Fourier dual (wave vector) $\xi$. For fixed frequency, $\omega$, we use the asymptotics as $\sqrt{\omega^2\hat{\mu}_0^{-1}-\xi^2}\to\infty$ of the Weyl matrix and the Jost function. We do require *two* distinct frequencies. Although the dependencies of Lamé parameters and required data are different from the inverse problem analyzed by Beals and Coifman,[11,12] various steps in our proofs follow the logic of their proofs. To keep the presentation self-contained, we will review certain aspects of the work of Beals, Henkin, and Novikova.

The main results of this paper are the following:

- We describe analytic properties of the Jost and Weyl solutions and functions for the Rayleigh Sturm–Liouville problem, in $\xi$, on the Riemann surface and study their asymptotic behaviors on the physical sheet (Sec. V).
- Using the Wronskian for the solutions of the adjoint Sturm-Liouville problems, we derive a formula representing the Weyl matrix in terms of the Jost function (Lemma V.2); we relate the Jost function to the boundary matrix for the original Rayleigh system (Appendix C).
- Following Ref. 8, we derive a Gel'fand-Levitan type equation relating the Weyl matrix to the apparent potential of the Rayleigh Sturm-Liouville equation (Proposition VII.2) and establish the uniqueness of its solution.
- We show the unique recovery of the Lamé parameters in two steps: Determining the Markushevich substitution at the bottom of the slab (Subsection VII B) and then recovering the potential (Subsection VII A) and the Lamé parameters in the slab using two distinct frequencies (Subsection VII C).

The Rayleigh system has been considered for many decades in seismology,[23] in particular, with the aim to estimate Lamé parameters from the observation of Rayleigh waves at a few frequencies.[13] Empirically, seismologists have found that the eigenvalues corresponding to these waves as the data are insufficient and have considered additional types of data in the absence of a mathematical understanding of this inverse problem. The fixed-frequency Rayleigh system is an extraction from the time-harmonic elastic-wave system of equations. The inverse boundary value problem for time-harmonic elastic waves on a bounded, Lipschitz subdomain of $\mathbb{R}^3$ has been studied before. Nakamura and Uhlmann[14,15] proved uniqueness, assuming that the Lamé parameters are $C^\infty$ and that the shear modulus is close to a positive constant. Eskin and Ralston[16] proved a related result. (In the context of the analyses of inverse boundary value problems for time-harmonic waves, we note that complex geometrical optics solutions employed in these are multidimensional generalizations of Jost solutions.) Beretta *et al.*[17] proved uniqueness and Lipschitz stability of this inverse problem when the Lamé parameters and the density are assumed to be piecewise constant on a given domain partition, with partial boundary data. Global uniqueness of the inverse problem in dimension three assuming general Lamé parameters remains an open problem. We note again that, here, the Lamé parameters only depend on the boundary normal coordinate.

## II. RAYLEIGH SYSTEM

We consider the Rayleigh operator associated with elastic surface Rayleigh waves in isotropic media,[9]

$$H_0(x,\xi)\begin{pmatrix}\varphi_1\\ \varphi_3\end{pmatrix}=\begin{pmatrix}-\frac{\partial}{\partial Z}\left(\hat{\mu}\frac{\partial\varphi_1}{\partial Z}\right)-\mathrm{i}|\xi|\left(\frac{\partial}{\partial Z}(\hat{\mu}\varphi_3)+\hat{\lambda}\frac{\partial}{\partial Z}\varphi_3\right)+(\hat{\lambda}+2\hat{\mu})|\xi|^2\varphi_1\\ -\frac{\partial}{\partial Z}\left((\hat{\lambda}+2\hat{\mu})\frac{\partial\varphi_3}{\partial Z}\right)-\mathrm{i}|\xi|\left(\frac{\partial}{\partial Z}(\hat{\lambda}\varphi_1)+\hat{\mu}\frac{\partial}{\partial Z}\varphi_1\right)+\hat{\mu}|\xi|^2\varphi_3\end{pmatrix}. \tag{1}$$

We will use the notation

$$\widetilde{w}=\begin{pmatrix}\varphi_1\\ \varphi_3\end{pmatrix}.$$

We denote the eigenvalues of $H_0(x,\xi)$ by $\Lambda_j=\Lambda_j(x,\xi)$. These follow from solving the following system ($Z<0$):

$$-\frac{\partial}{\partial Z}\hat{\mu}\frac{\partial\varphi_1}{\partial Z}-\mathrm{i}|\xi|\left(\frac{\partial}{\partial Z}(\hat{\mu}\varphi_3)+\hat{\lambda}\frac{\partial}{\partial Z}\varphi_3\right)+(\hat{\lambda}+2\hat{\mu})|\xi|^2\varphi_1=\Lambda\varphi_1, \tag{2}$$

$$-\frac{\partial}{\partial Z}(\hat{\lambda}+2\hat{\mu})\frac{\partial\varphi_3}{\partial Z}-\mathrm{i}|\xi|\left(\frac{\partial}{\partial Z}(\hat{\lambda}\varphi_1)+\hat{\mu}\frac{\partial}{\partial Z}\varphi_1\right)+\hat{\mu}|\xi|^2\varphi_3=\Lambda\varphi_3, \tag{3}$$

supplemented with the following Neumann boundary conditions:

$$a_-(\widetilde{w}):=\mathrm{i}\hat{\lambda}|\xi|\varphi_1(0^-)+(\hat{\lambda}+2\hat{\mu})\frac{\partial\varphi_3}{\partial Z}(0^-)=0, \tag{4}$$

$$b_-(\widetilde{w}):=\mathrm{i}|\xi|\hat{\mu}\varphi_3(0^-)+\hat{\mu}\frac{\partial\varphi_1}{\partial Z}(0^-)=0. \tag{5}$$

The square roots of the eigenvalues can be identified with the phase velocities. We will set $\rho \equiv 1$ and simplify the notation

$$\lambda = \hat{\lambda},\ \ \mu = \hat{\mu}\ \ (\rho = 1).$$

Rayleigh problems (2)–(5) correspond to (1.1), (1.1′) of Ref. 8 upon identifying

$$\begin{aligned} &x = -Z, \quad w_1 = -\mathrm{i}\varphi_1,\ \ w_2 = \varphi_3, \\ &\chi_1 = b_-(w) = \mathrm{i}b_-(\widetilde{w}),\ \ \chi_2 = a_-(w) = -a_-(\widetilde{w}), \quad \xi = |\xi|,\ \ \omega^2 = \Lambda. \end{aligned} \tag{6}$$

We proceed using this notation,[5,8]

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\mu\frac{\mathrm{d}w_1}{\mathrm{d}x} - \xi\mu w_2\right) - \xi\lambda\frac{\mathrm{d}w_2}{\mathrm{d}x} + \left(\omega^2 - \xi^2(\lambda + 2\mu)\right)w_1 = 0, \tag{7}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left((\lambda + 2\,\mu)\frac{\mathrm{d}w_2}{\mathrm{d}x} + \xi\lambda w_1\right) + \xi\mu\frac{\mathrm{d}w_1}{\mathrm{d}x} + (\omega^2 - \xi^2\mu)w_2 = 0, \tag{8}$$

supplemented with the following boundary conditions:

$$\left.\left(\mu\frac{\mathrm{d}w_1}{\mathrm{d}x} - \xi\mu w_2\right)\right|_{x=0^+} = \chi_1(\xi) = b_-(w), \tag{9}$$

$$\left.\left((\lambda + 2\mu)\frac{\mathrm{d}w_2}{\mathrm{d}x} + \xi\lambda w_1\right)\right|_{x=0^+} = \chi_2(\xi) = a_-(w). \tag{10}$$

We write $\chi = (\chi_1, \chi_2)^{\mathrm{T}}$ with $\chi = \chi(\xi)$.[18] From now on, to simplify notation, we will use $\xi$ for both $|\xi| \in \mathbb{R}_+$ and its values in $\mathbb{C}$ following analytic continuation.

We consider the case of an inhomogeneous isotropic elastic slab of thickness $H$ bonded to a homogeneous isotropic elastic half space with Lamé parameters $\lambda_0$ and $\mu_0$. We assume that the layer's Lamé parameters, $\lambda$ and $\mu$, are three times continuously differentiable and smoothly matched to the half-space constants, $\lambda_0$ and $\mu_0$, respectively. In earlier papers,[9] we used the notations $\lambda_I$ and $\mu_I$ for $\lambda_0$ and $\mu_0$, respectively, and $|Z_I| = H$.

*Assumption II.1. We let $\mu \geq \alpha_0 > 0$, $2\mu + 3\lambda \geq \beta_0 > 0$, and $\lambda, \mu \in C^3(\mathbb{R}_+)$ and $\lambda(x) = \lambda_0$, $\mu(x) = \mu_0$ for $x \geq H$.*

Assumption II.1 can be weakened to letting $\lambda \in C^1(\mathbb{R}_+)$. It reflects the *strong ellipticity* condition[19] as this appears in the existence and uniqueness of solutions of the boundary value problem for time-harmonic elastic waves. The parameters $\lambda$ and $\mu$ will be further restricted through Assumption V.2.

## III. MARKUSHEVICH TRANSFORM TO TWO ADJOINT MATRIX STURM-LIOUVILLE PROBLEMS

We perform an analog of the calibration transform on the Rayleigh system to obtain a matrix-valued (essentially non-diagonalizable) Sturm-Liouville problem. We follow Ref. 7.

Based on the Pekeris substitution,[6] it was shown by Markushevich[3–5] that the boundary value problem (7)–(10) with $\chi_1 = \chi_2 = 0$ can be reduced to two matrix Sturm-Liouville problems with mutually transposed potentials and boundary conditions. Here, we briefly outline the transformations for arbitrary $\chi_1$ and $\chi_2$. For conciseness of notation while suppressing the coordinate dual to $\xi$, in the remainder of the analysis, we use $'$ to denote differentiation with respect to $x$.

Let $G$ be a $2 \times 2$-matrix solving the Cauchy problem,

$$G' = \frac{1}{2}LG, \quad G(0) = I_2, \tag{11}$$

where $I_2$ is the unit matrix and

$$L = \begin{pmatrix} 0 & -d \\ -c & 0 \end{pmatrix} \quad \text{with} \quad c = \frac{1}{g_0}\frac{\mu(\lambda + \mu)}{(\lambda + 2\,\mu)},\ \ d = -2g_0\left(\frac{1}{\mu}\right)''. \tag{12}$$

We have $\det G(x) = 1$; cf. Ref. 3. We adopt the notation of Markushevich,[4] where $g_0$ stands for an arbitrary positive constant. It is convenient to put $g_0 = \mu_0$, which we do from now onward.

By the substitution ($x \in [0, \infty)$)

$$\mathfrak{M}^{-1}(F) = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \tag{13}$$

with

$$\mathfrak{M}^{-1} = \begin{pmatrix} \frac{\mathrm{d}}{\mathrm{d}x} & 1 \\ -\xi & 0 \end{pmatrix} \begin{pmatrix} \frac{\mu_0}{\mu} & 0 \\ 0 & \frac{\mu}{\lambda + 2\,\mu} \end{pmatrix} \left(G^{\mathrm{T}}\right)^{-1} \tag{14}$$

signifying the *inverse* Markushevich transform, problem (7)–(10) reduces to the following matrix Sturm-Liouville form:

$$F'' - \xi^2 F = QF, \qquad x \in (0,\infty), \tag{15}$$

$$F' + \Theta F = \left(D^{\mathrm{a}}\right)^{-1}\chi, \qquad x = 0. \tag{16}$$

Here, $\Theta = \Theta(\xi) = (D^{\mathrm{a}}(\xi))^{-1} C^{\mathrm{a}}(\xi)$ with

$$D^{\mathrm{a}}(\xi) = \begin{pmatrix} -2\mu_0 \frac{\mu'(0)}{\mu(0)} & \mu(0) \\ -2\mu_0\xi & 0 \end{pmatrix},$$

$$C^{\mathrm{a}}(\xi) = \begin{pmatrix} \mu_0\left(2\xi^2 - \frac{\omega^2}{\mu(0)} + \frac{\mu''(0)}{\mu(0)}\right) & -\frac{\mu'(0)\mu(0)}{\lambda(0) + 2\,\mu(0)} \\ 2\mu_0\xi\frac{\mu'(0)}{\mu(0)} & -\xi\frac{\mu^2(0)}{\lambda(0) + 2\,\mu(0)} \end{pmatrix} \tag{17}$$

so that

$$\left(D^{\mathrm{a}}(\xi)\right)^{-1} = \frac{1}{2\mu_0\mu\xi} \begin{pmatrix} 0 & -\mu(0) \\ 2\mu_0\xi & -2\mu_0\frac{\mu'(0)}{\mu(0)} \end{pmatrix} \tag{18}$$

and

$$\Theta(\xi) = \begin{pmatrix} -\frac{\mu'(0)}{\mu(0)} & \frac{1}{2\mu_0}\frac{\mu^2(0)}{(\lambda(0) + 2\,\mu(0))} \\ \frac{\mu_0}{\mu(0)}\left(2\xi^2 - \frac{\omega^2}{\mu(0)} - \mu(0)\left(\frac{1}{\mu}\right)''(0)\right) & 0 \end{pmatrix}$$

$$=: \begin{pmatrix} -\theta_3 & \theta_2 \\ 2\frac{\mu_0}{\mu(0)}\xi^2 - \theta_1 & 0 \end{pmatrix}. \tag{19}$$

Furthermore, $Q$ is the matrix-valued potential given by

$$Q = \left(G^{-1}BG\right)^{\mathrm{T}}, \qquad B = B_1 + \omega^2 B_2, \tag{20}$$

with

$$B_1 = \begin{pmatrix} -\frac{1}{2}\left(\frac{1}{\mu}\right)''\frac{\mu(\lambda+\mu)}{\lambda + 2\,\mu} + \frac{\mu''}{\mu} & \mu_0\left(2\frac{\mu'}{\mu}\left(\frac{1}{\mu}\right)'' + \left(\frac{1}{\mu}\right)'''\right) \\ \frac{1}{\mu_0}\left(\frac{\lambda'\mu^2 + \mu'\lambda(\lambda+\mu)}{(\lambda + 2\,\mu)^2} - \frac{1}{2}\left(\frac{\mu(\lambda+\mu)}{\lambda + 2\,\mu}\right)'\right) & \frac{1}{2}\left(\frac{1}{\mu}\right)''\frac{(\lambda-\mu)}{\lambda + 2\,\mu} \end{pmatrix}, \tag{21}$$

$$B_2 = \begin{pmatrix} -\frac{1}{\mu} & \mu_0\left(\frac{1}{\mu^2}\right)' \\ 0 & -\frac{1}{\lambda + 2\,\mu} \end{pmatrix}. \tag{22}$$

We note that the potential is not a symmetric matrix, that is, $Q \neq Q^{\mathrm{T}}$.

By the adjoint substitution

$$(\mathfrak{M}^{\mathrm{a}})^{-1}(F^{\mathrm{a}}) = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \tag{23}$$

with

$$(\mathfrak{M}^{\mathrm{a}})^{-1} = \begin{pmatrix} 0 & -\xi \\ 1 & \dfrac{\mathrm{d}}{\mathrm{d}x} \end{pmatrix} \begin{pmatrix} 1 & -2\mu_0 \left(\dfrac{1}{\mu}\right)' \\ 0 & \dfrac{\mu_0}{\mu} \end{pmatrix} G, \tag{24}$$

problem (7)–(10) transforms to the following matrix Sturm-Liouville form:

$$(F^{\mathrm{a}})'' - \xi^2 F^{\mathrm{a}} = Q^{\mathrm{a}} F^{\mathrm{a}}, \quad x \in (0, \infty), \tag{25}$$

$$(F^{\mathrm{a}})' + \Theta^{\mathrm{a}} F^{\mathrm{a}} = D^{-1}\chi, \quad x = 0. \tag{26}$$

Here,

$$Q^{\mathrm{a}} = Q^{\mathrm{T}}, \quad \Theta^{\mathrm{a}} = \Theta^{\mathrm{T}} = \Theta^{\mathrm{T}}(\xi) = D^{-1}(\xi) C(\xi) \tag{27}$$

is a $2 \times 2$-matrix with $D(\xi)$ and $C(\xi)$ being the matrices

$$D(\xi) = \begin{pmatrix} 0 & -2\xi\mu_0 \\ \mu(0) & 0 \end{pmatrix},$$
$$C(\xi) = \begin{pmatrix} -\xi \dfrac{\mu^2(0)}{(\lambda(0) + 2\,\mu(0))} & 0 \\ -\mu'(0) & \dfrac{\mu_0}{\mu(0)} \left(2\,\mu(0)\xi^2 - \omega^2 - 2\dfrac{(\mu'(0))^2}{\mu(0)} + \mu''(0)\right) \end{pmatrix}. \tag{28}$$

### A. Homogeneous half space, $x \in (H, \infty)$

In components, (11) has the form

$$G_{11}' = -\frac{d}{2} G_{21}, \quad G_{12}' = -\frac{d}{2} G_{22}, \quad G_{21}' = -\frac{c}{2} G_{11}, \quad G_{22}' = -\frac{c}{2} G_{12}, \tag{29}$$

in which, in view of (12), the coefficient $d$ is zero if $\mu$ is constant. We consider the (homogeneous) half space $x \in (H, \infty)$ and write

$$G_{11}(H) = G_{11}^H, \quad G_{12}(H) = G_{12}^H, \quad G_{21}(H) = G_{21}^H, \quad G_{22}(H) = G_{22}^H. \tag{30}$$

Then, the matrix function, $G$, inside $x \in (H, \infty)$ can be determined from the Cauchy problem,

$$G' = -\frac{c_0}{2} \begin{pmatrix} 0 & 0 \\ G_{11} & G_{12} \end{pmatrix}, \quad G(H) = \begin{pmatrix} G_{11}^H & G_{12}^H \\ G_{21}^H & G_{22}^H \end{pmatrix}, \tag{31}$$

in which

$$c_0 = \frac{\lambda_0 + \mu_0}{\lambda_0 + 2\mu_0}. \tag{32}$$

The solution is given as

$$\begin{gathered} G_{11}(x) = G_{11}^H, \quad G_{12}(x) = G_{12}^H, \\ G_{21}(x) = -\frac{c_0}{2} G_{11}^H (x - H) + G_{21}^H, \quad G_{22}(x) = -\frac{c_0}{2} G_{12}^H (x - H) + G_{22}^H. \end{gathered} \tag{33}$$

As $\det G(x) = 1$ (see Refs. 3–5), the inverse matrix follows to be

$$G^{-1} = \begin{pmatrix} G_{22} & -G_{12} \\ -G_{21} & G_{11} \end{pmatrix}. \tag{34}$$

Thus, in the homogeneous elastic half space, $x \in (H, \infty)$, according to (20)–(22) and (34), we have

$$Q = \omega^2 \begin{pmatrix} \frac{G_{12}G_{21}}{\lambda_0 + 2\mu_0} - \frac{G_{11}G_{22}}{\mu_0} & G_{11}G_{21}\frac{c_0}{\mu_0} \\ -G_{12}G_{22}\frac{c_0}{\mu_0} & \frac{G_{12}G_{21}}{\mu_0} - \frac{G_{11}G_{22}}{\lambda_0 + 2\mu_0} \end{pmatrix}, \tag{35}$$

where the components of the transformation matrix $G$ are given by (33). It is of interest to observe that if $G_{12}^H \neq 0$, then all components of the potential matrix, $Q$, will have linear growth as $x \to \infty$.

From here onward, we denote $Q(x)$ for $x \geq H$ by $Q_0(x)$. Using, again, that $\det G(x) = 1$, we obtain

$$\begin{aligned} Q_0(x) &= \omega^2 \begin{pmatrix} -\frac{1}{\mu_0} & 0 \\ 0 & -\frac{1}{\lambda_0 + 2\mu_0} \end{pmatrix} + \omega^2 \frac{\lambda_0 + \mu_0}{\mu_0(\lambda_0 + 2\mu_0)} \begin{pmatrix} -G_{12}G_{21} & G_{21}G_{11} \\ -G_{12}G_{22} & G_{12}G_{21} \end{pmatrix} \\ &= \omega^2 \begin{pmatrix} -\frac{1}{\mu_0} & 0 \\ 0 & -\frac{1}{\lambda_0 + 2\mu_0} \end{pmatrix} \\ &\quad + \omega^2 \frac{c_0}{\mu_0} \begin{pmatrix} -G_{12}^H\left[-\frac{c_0}{2}G_{11}^H(x-H) + G_{21}^H\right] & G_{11}^H\left[-\frac{c_0}{2}G_{11}^H(x-H) + G_{21}^H\right] \\ -G_{12}^H\left[-\frac{c_0}{2}G_{12}^H(x-H) + G_{22}^H\right] & G_{12}^H\left[-\frac{c_0}{2}G_{11}^H(x-H) + G_{21}^H\right] \end{pmatrix}. \end{aligned} \tag{36}$$

We extend $Q_0 = Q_0(x)$ to $x \in (0, H]$ linear in $x$ and refer to it as the background potential. Then, we introduce the perturbation potential

$$V(x) = Q(x) - Q_0(x)$$

so that $V(x) = 0$ for $x \geq H$.

*Remark III.1. If $H = 0$, then $G_{12}^H = G_{21}^H = 0$, $G_{11}^H = G_{22}^H = 1$, and*

$$Q_0(x) = \omega^2 \begin{pmatrix} -\frac{1}{\mu_0} & 0 \\ 0 & -\frac{1}{\lambda_0 + 2\mu_0} \end{pmatrix} + \omega^2 \frac{c_0^2}{2\mu_0} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x.$$

We write

$$\varpi = \frac{\mu_0}{\mu(0)}$$

and introduce a class of potentials.

*Definition III.1. A real matrix-valued potential, $Q$, is of Lamé type if it can be generated from Lamé parameters according to the Markushevich transform, that is, of the form (20)–(22). According to Assumption II.1, $Q \in C^1(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+)$.*

In view of Assumption II.1, our matrix-valued potential, $Q$, of Lamé type attains the form $Q_0$ on $[H, \infty)$, as in (36). Then, $V = Q - Q_0 \in L^1([0, H])$.

The Lamé parameters at $x = 0$ and $x \geq H$, that is, $\lambda(0)$, $\mu(0)$ *and* $\mu'(0)$, $\mu''(0)$ *and* $\lambda_0$ *and* $\mu_0$*, are encoded in and determine* $\Theta$ *(or* $\theta_1$*,* $\theta_2$*, and* $\theta_3$*)* independently of $Q$. In this paper, we will not consider the problem of boundary determination.

## IV. JOST AND WEYL SOLUTIONS, JOST FUNCTION AND WEYL MATRIX

We introduced $V$ in (15) and obtain

$$-F'' + Q_0 F + VF = -\xi^2 F, \quad x \in (0, \infty), \tag{37}$$

supplemented with (16),

$$F' + \Theta F = \left(D^a\right)^{-1}\chi, \quad x = 0, \tag{38}$$

where $\Theta$ is given in (19).

First, we construct solutions to the "background" equation

$$-F'' + Q_0 F = -\xi^2 F \tag{39}$$

of the form

$$F^{\pm}_{P,0} = \begin{pmatrix} F^{\pm}_{P,0,1} \\ F^{\pm}_{P,0,2} \end{pmatrix} e^{\pm \mathrm{i} x q_P}, \quad q_P = \sqrt{\frac{\omega^2}{\lambda_0 + 2\mu_0} - \xi^2},$$
$$F^{\pm}_{S,0} = \begin{pmatrix} F^{\pm}_{S,0,1} \\ F^{\pm}_{S,0,2} \end{pmatrix} e^{\pm \mathrm{i} x q_S}, \quad q_S = \sqrt{\frac{\omega^2}{\mu_0} - \xi^2}$$

so that their inverse Markushevich transforms [$\mathfrak{M}^{-1}(F^{\pm}_{P,0}), \mathfrak{M}^{-1}(F^{\pm}_{S,0})$; cf. (14)] are proportional to

$$\begin{pmatrix} 1 \\ \mp \frac{\mathrm{i}}{\xi} q_P \end{pmatrix} e^{\pm \mathrm{i} x q_P}, \quad \begin{pmatrix} \mp \frac{\mathrm{i}}{\xi} q_S \\ 1 \end{pmatrix} e^{\pm \mathrm{i} x q_S}, \tag{40}$$

respectively. We make this precise below. We refer to $q_P$ and $q_S$ as quasi-momenta. We note that (40) are solutions to Rayleigh system (7)-(8) with $\lambda = \lambda_0$, $\mu = \mu_0$ constant for all $x \geq 0$. We may construct similar solutions to the adjoint equation

$$-(F^{\mathrm{a}})'' + Q_0^{\mathrm{T}} F^{\mathrm{a}} = -\xi^2 F^{\mathrm{a}}. \tag{41}$$

We consider (15) and (25) on $x \in (H, \infty)$, where the potential $Q$ is given by formulas (35) and (36) with the transformation matrix $G$ determined by (33). Using that

$$\begin{pmatrix} \frac{\mathrm{d}}{\mathrm{d}x} & 1 \\ -\xi & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & -\frac{1}{\xi} \\ 1 & \frac{1}{\xi}\frac{\mathrm{d}}{\mathrm{d}x} \end{pmatrix}, \quad \begin{pmatrix} 0 & -\xi \\ 1 & \frac{\mathrm{d}}{\mathrm{d}x} \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{\xi}\frac{\mathrm{d}}{\mathrm{d}x} & 1 \\ -\frac{1}{\xi} & 0 \end{pmatrix},$$

(14), for $x \in (H, \infty)$, implies that

$$F = \mathfrak{M}(w) = G^{\mathrm{T}} \begin{pmatrix} 1 & 0 \\ 0 & \frac{\lambda_0 + 2\mu_0}{\mu_0} \end{pmatrix} \begin{pmatrix} 0 & -\frac{1}{\xi} \\ 1 & \frac{1}{\xi}\frac{\mathrm{d}}{\mathrm{d}x} \end{pmatrix} w, \tag{42}$$

while (24), for $x \in (H, \infty)$, implies that

$$F^{\mathrm{a}} = \mathfrak{M}^{\mathrm{a}}(w) = G^{-1} \begin{pmatrix} \frac{1}{\xi}\frac{\mathrm{d}}{\mathrm{d}x} & 1 \\ -\frac{1}{\xi} & 0 \end{pmatrix} w, \tag{43}$$

where $w = (w_1, w_2)^{\mathrm{T}}$ solves the Rayleigh system (7)-(8). For $x \in (H, \infty)$, we have linearly independent solutions

$$w^{\pm}_{P,0} = \mu_0 \frac{\xi^2}{\omega^2} \begin{pmatrix} 1 \\ \mp \frac{\mathrm{i}}{\xi} q_P \end{pmatrix} e^{\pm \mathrm{i} x q_P}, \tag{44}$$

$$w^{\pm}_{S,0} = \mu_0 \frac{\xi^2}{\omega^2} \begin{pmatrix} \mp \frac{\mathrm{i}}{\xi} q_S \\ 1 \end{pmatrix} e^{\pm \mathrm{i} x q_S}, \tag{45}$$

which correspond to the solutions of the original Rayleigh system (2)-(3),

$$\widetilde{w}^{\mp}_{P,0} = \mathrm{i}\mu_0 \frac{\xi}{\omega^2} \begin{pmatrix} \xi \\ \mp q_P \end{pmatrix} e^{\mp i Z q_P} =: \mathrm{i}\mu_0 \frac{\xi}{\omega^2} f^{\mp}_{P,0}, \tag{46}$$

$$\widetilde{w}^{\mp}_{S,0} = -\mu_0 \frac{\xi}{\omega^2} \begin{pmatrix} \mp q_S \\ -\xi \end{pmatrix} e^{\mp iZq_S} =: -\mu_0 \frac{\xi}{\omega^2} f^{\mp}_{S,0}. \tag{47}$$

Using (33), substitution of (44) and (45) into (42) and (43) yields

$$F^{\pm}_{P,0} = \mathfrak{M}(w^{\pm}_{P,0}) = \begin{pmatrix} -\frac{c_0}{2} G^H_{11}(x-H) + G^H_{21} \pm \mathrm{i} q_P \frac{\mu_0}{\omega^2} G^H_{11} \\ -\frac{c_0}{2} G^H_{12}(x-H) + G^H_{22} \pm \mathrm{i} q_P \frac{\mu_0}{\omega^2} G^H_{12} \end{pmatrix} e^{\pm \mathrm{i} x q_P}, \tag{48}$$

$$F^{\pm}_{S,0} = \mathfrak{M}(w^{\pm}_{S,0}) = -\mu_0 \frac{\xi}{\omega^2} \begin{pmatrix} G^H_{11} \\ G^H_{12} \end{pmatrix} e^{\pm \mathrm{i} x q_S} \tag{49}$$

and

$$F^{\mathrm{a},\pm}_{S,0} = \mathfrak{M}^{\mathrm{a}}(w^{\pm}_{S,0}) = \begin{pmatrix} -\frac{c_0}{2} G^H_{12}(x-H) + G^H_{22} \mp \mathrm{i} q_S \frac{\mu_0}{\omega^2} G^H_{12} \\ \frac{c_0}{2} G^H_{11}(x-H) - G^H_{21} \pm \mathrm{i} q_S \frac{\mu_0}{\omega^2} G^H_{11} \end{pmatrix} e^{\pm \mathrm{i} x q_S}, \tag{50}$$

$$F^{\mathrm{a},\pm}_{P,0} = \mathfrak{M}^{\mathrm{a}}(w^{\pm}_{P,0}) = \mu_0 \frac{\xi}{\omega^2} \begin{pmatrix} G^H_{12} \\ -G^H_{11} \end{pmatrix} e^{\pm \mathrm{i} x q_P}. \tag{51}$$

*Remark IV.1. If $H = 0$, substituting $G^H_{12} = G^H_{21} = 0$ and $G^H_{11} = G^H_{22} = 1$ into (48)-(49) and (50)-(51), respectively, yields*

$$F^{\pm}_{P,0} = \begin{pmatrix} -\frac{c_0}{2} x \pm \mathrm{i} q_P \frac{\mu_0}{\omega^2} \\ 1 \end{pmatrix} e^{\pm \mathrm{i} x q_P}, \quad F^{\pm}_{S,0} = \begin{pmatrix} -\mu_0 \frac{\xi}{\omega^2} \\ 0 \end{pmatrix} e^{\pm \mathrm{i} x q_S}$$

*and*

$$F^{\mathrm{a},\pm}_{S,0} = \begin{pmatrix} 1 \\ \frac{c_0}{2} x \pm \mathrm{i} q_S \frac{\mu_0}{\omega^2} \end{pmatrix} e^{\pm \mathrm{i} x q_S}, \quad F^{\mathrm{a},\pm}_{P,0} = \begin{pmatrix} 0 \\ -\mu_0 \frac{\xi}{\omega^2} \end{pmatrix} e^{\pm \mathrm{i} x q_P}.$$

In view of the presence of square roots, $q_P$ and $q_S$, we introduce the cut complex plane,

$$\mathscr{K} = \mathbb{C} \backslash \left( \left[ -\frac{\omega}{\sqrt{\mu_0}}, \frac{\omega}{\sqrt{\mu_0}} \right] \cup \mathrm{i}\mathbb{R} \right).$$

In Appendix A, we introduce the corresponding Riemann surface and physical Riemann sheet, $\mathscr{K}_+$, by the condition $\operatorname{Im} q_P(\xi) > 0$, $\operatorname{Im} q_S(\xi) > 0$.

The Jost solutions, $F^{\pm}_P, F^{\pm}_S$, of (37) are determined by the conditions

$$F^{\pm}_P = F^{\pm}_{P,0}, \quad F^{\pm}_S = F^{\pm}_{S,0} \qquad \text{for} \quad x \geq H.$$

We define the matrix Jost solutions, $\mathbf{F} = \mathbf{F}(x,\xi)$ and $\mathbf{F}_0 = \mathbf{F}_0(x,\xi)$ (emphasizing that, here, $\xi$ is *not* the Fourier dual to $x$), as

$$\mathbf{F}(x,\xi) = [F^+_P \ F^+_S], \quad \mathbf{F}_0(x,\xi) = [F^+_{P,0} \ F^+_{S,0}], \tag{52}$$

and we define the Jost function as [cf. (16)]

$$\mathbf{F}_{\Theta}(\xi) = \mathbf{F}'(0,\xi) + \Theta(\xi)\mathbf{F}(0,\xi),$$

where $\Theta(\xi)$ is given in (19).

*Remark IV.2. By (C6), we have* $\det \mathbf{F}_{\Theta}(\xi) = c\xi\Delta_{\mathrm{R}}$*, where $\Delta_{\mathrm{R}}$ is the Rayleigh determinant (see Remark IV.3) and c is a constant. The Rayleigh determinant originates from the reflection matrix that we will introduce later.*

The Jost solutions, $F_P^{a,\pm}$, $F_S^{a,\pm}$, of

$$-(F^a)'' + Q_0^T F^a + V^T F^a = -\xi^2 F^a, \quad x \in (0,\infty) \tag{53}$$

[upon introducing $V$ in (25)], are determined by the following conditions:

$$F_P^{a,\pm} = F_{P,0}^{a,\pm}, \quad F_S^{a,\pm} = F_{S,0}^{a,\pm} \qquad \text{for} \quad x > H.$$

We define the matrix Jost solutions, $\mathbf{F}^a = \mathbf{F}^a(x,\xi)$ and $\mathbf{F}_0^a = \mathbf{F}_0^a(x,\xi)$, as

$$\mathbf{F}^a(x,\xi) = [F_P^{a,+} \; F_S^{a,+}], \quad \mathbf{F}_0^a(x,\xi) = [F_{P,0}^{a,+} \; F_{S,0}^{a,+}], \tag{54}$$

and we define the Jost function as [cf. (26)]

$$\mathbf{F}_\Theta^a(\xi) = (\mathbf{F}^a)'(0,\xi) + \Theta^a(\xi)\mathbf{F}^a(0,\xi).$$

Using (C6) and (C7), we find that

$$\mathbf{F}_\Theta^a(\xi) = \begin{pmatrix} -2\dfrac{\mu_0}{\mu(0)}\xi & 0 \\ \dfrac{\mu'(0)}{\mu(0)}\dfrac{1}{\xi} & -\dfrac{\mu(0)}{2\mu_0}\dfrac{1}{\xi} \end{pmatrix} \mathbf{F}_\Theta(\xi). \tag{55}$$

We note that $\mathbf{F}^a = (\mathbf{F}^\star)^T$, where $\mathbf{F}^\star$ denotes the solution to the adjoint problem according to Ref. 2. The Wronskian of the Jost solutions of the adjoint problems has the familiar property of being independent of $x$,

$$\frac{d}{dx} W(\mathbf{F}^a, \mathbf{F}) = 0. \tag{56}$$

We obtain the following.

*Lemma IV.1. Let* $\mathbf{F}$, $\mathbf{F}_0$, $\mathbf{F}^a$, *and* $\mathbf{F}_0^a$ *be given by (52) and (54), respectively. Then,*

$$\begin{aligned} W(\mathbf{F}^a(x,-\xi), \mathbf{F}(x,\xi)) &= W(\mathbf{F}_0^a(x,-\xi), \mathbf{F}_0(x,\xi)) \\ &= ((\mathbf{F}_0^a)')^T(x,-\xi)\mathbf{F}_0(x,\xi) - (\mathbf{F}_0^a)^T(x,-\xi)\mathbf{F}_0'(x,\xi) = -\mathrm{i}\,2\mu_0\frac{\xi}{\omega^2}\begin{pmatrix} q_P & 0 \\ 0 & -q_S \end{pmatrix}. \end{aligned} \tag{57}$$

Now, we define the Weyl solution, $\mathbf{\Phi}$, as

$$\mathbf{\Phi}(x,\xi) = \mathbf{F}(x,\xi)[\mathbf{F}_\Theta(\xi)]^{-1} \tag{58}$$

and the Weyl matrix, $\mathbf{M}$, as

$$\mathbf{M}(\xi) = \mathbf{\Phi}(0,\xi) = \mathbf{F}(0,\xi)[\mathbf{F}_\Theta(\xi)]^{-1}. \tag{59}$$

This definition shows that $\mathbf{M}(\xi)\mathbf{F}_\Theta(\xi) = \mathbf{F}(0,\xi)$, whence $\mathbf{M}(\xi)$ can be identified with the **Robin-to-Dirichlet map** associated with the matrix Sturm-Liouville problem (15). Clearly, also

$$\mathbf{\Phi}(x,\xi) = \mathbf{F}(x,\xi)[\mathbf{F}(0,\xi)]^{-1}\mathbf{M}(\xi). \tag{60}$$

*Remark IV.3. In a homogeneous half space, explicit calculations result in*

$$\mathbf{M}(\xi) = \frac{\mu_0\xi}{\omega^2\Delta_0(\xi)}\frac{1}{\mathrm{i}}\begin{pmatrix} \mathrm{i}q_P & \dfrac{1}{2} - \dfrac{\mu_0}{\omega^2}\xi^2 - \dfrac{\mu_0}{\omega^2}q_P q_S \\ \dfrac{\omega^2}{\mu_0} - 2\xi^2 & \mathrm{i}q_S \end{pmatrix}, \tag{61}$$

*where* $\Delta_0 = \det \mathbf{F}_{0,\Theta} = -\frac{\mu_0^2}{2\omega^4}\xi\Delta_R$ *with [cf. (46) and (47)]*

$$\Delta_R(\xi) = \left(\left(\frac{\omega^2}{\mu_0} - 2\xi^2\right)^2 + 4\xi^2 q_P q_S\right) = -\frac{1}{\mu_0^2}\det\begin{pmatrix} a_-(f_{P,0}^-) & a_-(f_{S,0}^-) \\ b_-(f_{P,0}^-) & b_-(f_{S,0}^-) \end{pmatrix}. \tag{62}$$

We have

$$\boldsymbol{\Phi}'(0,\xi) + \Theta(\xi)\boldsymbol{\Phi}(0,\xi) = I_2. \tag{63}$$

In a similar fashion, we introduce

$$\boldsymbol{\Phi}^{\mathrm{a}}(x,\xi) = \mathbf{F}^{\mathrm{a}}(x,\xi)[\mathbf{F}^{\mathrm{a}}_{\Theta}(\xi)]^{-1} \tag{64}$$

and the Weyl matrix, $\mathbf{M}^{\mathrm{a}} = \mathbf{M}^{\mathrm{a}}(\xi)$, as

$$\mathbf{M}^{\mathrm{a}}(\xi) = \boldsymbol{\Phi}^{\mathrm{a}}(0,\xi) = \mathbf{F}^{\mathrm{a}}(0,\xi)[\mathbf{F}^{\mathrm{a}}_{\Theta}(\xi)]^{-1}.$$

We have

$$(\boldsymbol{\Phi}^{\mathrm{a}})'(0,\xi) + \Theta^{\mathrm{a}}(\xi)\boldsymbol{\Phi}(0,\xi) = I_2. \tag{65}$$

We make the following observation. Using (70) and (75), we evaluate the Wronskian

$$W(\boldsymbol{\Phi}^{\mathrm{a}},\boldsymbol{\Phi}) = W(\boldsymbol{\Phi}^{\mathrm{a}},\boldsymbol{\Phi})\big|_{x=0} = \mathbf{M} - (\mathbf{M}^{\mathrm{a}})^{\mathrm{T}}. \tag{66}$$

As, using the expressions for $\mathbf{F}_0$ and $\mathbf{F}^{\mathrm{a}}_0$ and independence of the Wronskian of $x$,

$$\lim_{x\to\infty} W(\boldsymbol{\Phi}^{\mathrm{a}},\boldsymbol{\Phi}) = 0 \quad \text{for } \xi \in \mathscr{K}_+,$$

we conclude that

$$\mathbf{M}^{\mathrm{a}} = \mathbf{M}^{\mathrm{T}}. \tag{67}$$

## A. Other solutions

Following Ref. 2, we introduce two other matrix-valued solutions, $\boldsymbol{\varphi}(x,\xi)$, $\mathbf{S}(x,\xi)$, of (15), that is, solutions to

$$-\mathbf{F}'' + Q\mathbf{F} = -\xi^2\mathbf{F}, \tag{68}$$

satisfying

$$\boldsymbol{\varphi}(0,\xi) = I_2,\ \boldsymbol{\varphi}'(0,\xi) = -\Theta(\xi), \quad \mathbf{S}(0,\xi) = \mathbf{0},\ \mathbf{S}'(0,\xi) = I_2.$$

Hence, $\boldsymbol{\varphi}$ satisfies the Robin-type boundary condition

$$\mathbf{F}' + \Theta\mathbf{F} = 0, \quad x = 0. \tag{69}$$

Then, the Weyl solution takes the form

$$\boldsymbol{\Phi}(x,\xi) = \mathbf{S}(x,\xi) + \boldsymbol{\varphi}(x,\xi)\mathbf{M}(\xi). \tag{70}$$

Furthermore, we introduce the Green's function, $\mathfrak{G} = \mathfrak{G}(x,y)$, $0 < x < y$, satisfying

$$-\mathfrak{G}'' + Q_0\mathfrak{G} = -\xi^2\mathfrak{G}, \tag{71}$$

supplemented with

$$\mathfrak{G}(y,y) = \mathbf{0}, \quad \frac{\partial}{\partial x}\mathfrak{G}(x,y)\Big|_{x=y} = I_2.$$

See Appendix B for explicit expressions for $\mathfrak{G}$. From the definition, it follows that the Green's function is entire in $\xi \in \mathbb{C}$ (for any $\omega \in \mathbb{C}$ fixed). The matrix Jost solution $\mathbf{F}(x,\xi)$ then satisfies the Volterra-type integral equation,

$$\mathbf{F}(x,\xi) = \mathbf{F}_0(x,\xi) - \int_x^H \mathfrak{G}(x,y)V(y)\mathbf{F}(y,\xi)\,\mathrm{d}y. \tag{72}$$

In a similar fashion, we introduce two other matrix-valued solutions, $\boldsymbol{\varphi}^{\mathrm{a}}(x,\xi)$, $\mathbf{S}^{\mathrm{a}}(x,\xi)$, of (25), that is, solutions to

$$-(\mathbf{F}^{\mathrm{a}})'' + Q^{\mathrm{a}}\mathbf{F}^{\mathrm{a}} = -\xi^2\mathbf{F}^{\mathrm{a}}, \tag{73}$$

satisfying

$$\boldsymbol{\varphi}^{\mathrm{a}}(0,\xi) = I_2,\ (\boldsymbol{\varphi}^{\mathrm{a}})'(0,\xi) = -\Theta^{\mathrm{a}}(\xi), \quad \mathbf{S}^{\mathrm{a}}(0,\xi) = \mathbf{0},\ (\mathbf{S}^{\mathrm{a}})'(0,\xi) = I_2.$$

Hence, $\boldsymbol{\varphi}^{\mathrm{a}}$ satisfies the Robin-type boundary condition,

$$(\mathbf{F}^{\mathrm{a}})' + \Theta^{\mathrm{a}}\mathbf{F}^{\mathrm{a}} = 0, \quad x = 0. \tag{74}$$

Then, the Weyl solution takes the form

$$\boldsymbol{\Phi}^{\mathrm{a}}(x,\xi) = \mathbf{S}^{\mathrm{a}}(x,\xi) + \boldsymbol{\varphi}^{\mathrm{a}}(x,\xi)\mathbf{M}^{\mathrm{a}}(\xi). \tag{75}$$

Furthermore, we introduce the Green's function, $\mathfrak{G}^{\mathrm{a}} = \mathfrak{G}^{\mathrm{a}}(x,y)$, $0 < x < y$, satisfying

$$-(\mathfrak{G}^{\mathrm{a}})'' + Q_0^{\mathrm{T}}\mathfrak{G}^{\mathrm{a}} = -\xi^2\mathfrak{G}^{\mathrm{a}}, \tag{76}$$

supplemented with

$$\mathfrak{G}^{\mathrm{a}}(y,y) = \mathbf{0}, \qquad \frac{\partial}{\partial x}\mathfrak{G}^{\mathrm{a}}(x,y)\Big|_{x=y} = I_2.$$

The matrix Jost solution $\mathbf{F}^{\mathrm{a}}(x,\xi)$ then satisfies the Volterra-type integral equation,

$$\mathbf{F}^{\mathrm{a}}(x,\xi) = \mathbf{F}_0^{\mathrm{a}}(x,\xi) - \int_x^H \mathfrak{G}^{\mathrm{a}}(x,y)[V(y)]^{\mathrm{T}}\mathbf{F}^{\mathrm{a}}(y,\xi)\,\mathrm{d}y. \tag{77}$$

## V. SPECTRAL DATA

**Analytic continuation**

We note that

$$q_S(-\xi) = -q_S(\xi), \quad \xi \in \mathscr{K},$$

with an extension to the branch cuts,

$$\left(\left[-\frac{\omega}{\sqrt{\mu_0}}, \frac{\omega}{\sqrt{\mu_0}}\right] \cup \mathrm{i}\mathbb{R}\right).$$

See Appendix A. We give conjugation properties of the matrix Jost solutions in the following.

*Lemma V.1. For $\xi \in \mathscr{K} = \mathbb{C}\backslash\left(\left[-\frac{\omega}{\sqrt{\mu_0}}, \frac{\omega}{\sqrt{\mu_0}}\right] \cup \mathrm{i}\mathbb{R}\right)$ (see Appendix A), the following holds true:*

$$\overline{\mathbf{F}(x,\xi)} = \mathbf{F}(x,\bar{\xi}), \quad \overline{\mathbf{F}^{\mathrm{a}}(x,\xi)} = \mathbf{F}^{\mathrm{a}}(x,\bar{\xi}), \tag{78}$$

$$\mathbf{F}(x,-\xi) = [F_P^+(x,-\xi)\ F_S^+(x,-\xi)] = [F_P^-(x,\xi)\ \ -F_S^-(x,\xi)], \tag{79}$$

$$\mathbf{F}^{\mathrm{a}}(x,-\xi) = [F_P^{\mathrm{a},+}(x,-\xi)\ F_S^{\mathrm{a},+}(x,-\xi)] = [-F_P^{\mathrm{a},-}(x,\xi)\ F_S^{\mathrm{a},-}(x,\xi)]. \tag{80}$$

*Proof.* These properties are satisfied by the reference Jost solutions $\mathbf{F}_0$ and $\mathbf{F}_0^{\mathrm{a}}$. Then, we use the Volterra-type integral equations (72), (77) and the properties of the kernels (as even functions in both $q_S$ and $q_P$) and identities (A3). □

The conjugation properties in Lemma V.1 also imply that

$$\boldsymbol{\Phi}(\xi) = \overline{\boldsymbol{\Phi}(\bar{\xi})} \tag{81}$$

and

$$\mathbf{M}(\xi) = \overline{\mathbf{M}(\bar{\xi})} \tag{82}$$

on $\mathscr{K}_+$.

We observe that $\mathbf{M}$ has a meromorphic continuation from the physical ("upper") sheet $\mathscr{K}_+$ through the cuts to the unphysical ("lower") sheets and whole Riemann surface $\mathscr{R}$ (see Appendix A), which still satisfies this conjugation property.

We will simplify the analysis in Subsection V A by introducing

$$\mathscr{K}_+ \to \Pi_+, \quad \xi \to \zeta = \xi^2,$$

below, and note that this also defines the inverse $\Pi_+ \ni \zeta \to \xi = \sqrt{\zeta} \in \mathscr{K}_+$.

### A. Cauchy integral

From asymptotic expansion of the Weyl matrix that we develop in Lemma VI.1, specifically (109), it follows that

$$\det\mathbf{M}(\xi) = \frac{1}{\xi^2}\frac{\lambda(0) + 2\,\mu(0)}{\lambda(0) + \mu(0)} + \mathscr{O}\left(\frac{1}{\xi^3}\right) \qquad \text{as} \quad |\xi| \to \infty,\ \xi \in \mathscr{K}_+,$$

which implies that $\mathbf{M}$ has a finite number, $N$, say, of poles $\xi_j \in \mathscr{K}_+$. Here, $N$ depends on $\omega$. As $q_S(\xi_j) \in \mathrm{i}\mathbb{R}_+$, the poles are necessarily real. Moreover, $\mathbf{M}$ has no other poles in $\mathscr{K}_+$. We order the set of poles of the Weyl matrix

$$\frac{\omega}{\sqrt{\mu_0}} < \xi_N < \xi_{N-1} < \cdots < \xi_1$$

and invoke the following.

*Assumption V.1. The poles of* $\mathbf{M}$ *in* $\mathscr{K}_+$ *are simple.*

*Remark V.1. The poles of* $\mathbf{M}$ *are among the zeros of the determinant of the Jost function* $\det \mathbf{F}_\Theta$ *or Rayleigh determinant on the physical sheet,* $\mathscr{K}_+$*, and correspond to the bound states or normal modes. Assumption V.1 is generically satisfied for all frequencies* $\omega \in \mathbb{R}_+$*, that is, except possibly for a finite number of frequencies. In the seismology literature, the Rayleigh system is usually considered on a finite slab with traction-free boundary conditions when simplicity of the normal modes is well known.*[20] *In view of our outgoing radiation boundary condition at one end represented by (98), this result does not directly apply.*

We associate "energies" with the mentioned poles, $\zeta_j = \xi_j^2 \in \Pi_+$ [cf. (A4)]. We may introduce $\widehat{\mathbf{M}} = \widehat{\mathbf{M}}(\zeta)$ by

$$\widehat{\mathbf{M}}(\zeta(\xi)) = \mathbf{M}(\xi),$$

which thus has simple poles $\zeta_1, \ldots, \zeta_N$. We write $\Lambda' = \{\zeta_j\}_{j=1}^N$ and note that $\zeta_j > \frac{\omega^2}{\mu_0}$.

*Lemma V.2. The matrix* $\widehat{\mathbf{M}}$ *is analytic in* $\Pi_+$ *outside* $\Lambda'$*. It admits the representation*

$$\widehat{\mathbf{M}}(\zeta) = \int_{-\infty}^{\frac{\omega^2}{\mu_0}} \frac{\widehat{\mathbf{T}}(\eta)}{\zeta - \eta} \mathrm{d}\eta + \sum_{j=1}^N \frac{\alpha_j}{\zeta - \zeta_j}, \quad \zeta \in \Pi_+ \backslash \Lambda', \tag{83}$$

*where*

$$\alpha_j = \mathrm{Res}_{\zeta=\zeta_j} \widehat{\mathbf{M}}(\zeta) = \mathbf{F}(0, \xi_j) u_j, \quad u_j = 2\xi_j \mathrm{Res}_{\xi=\xi_j} [\mathbf{F}_\Theta(\xi)]^{-1} \tag{84}$$

*or*

$$\alpha_j = -[u_j^{\mathrm{a}}]^{\mathrm{T}} \int_0^\infty [\mathbf{F}^{\mathrm{a}}(x, \xi_j)]^{\mathrm{T}} \mathbf{F}(x, \xi) \mathrm{d}x\, u_j, \quad u_j^{\mathrm{a}} = 2\xi_j \mathrm{Res}_{\xi=\xi_j} [\mathbf{F}_\Theta^{\mathrm{a}}(\xi)]^{-1} \tag{85}$$

*or*

$$\alpha_j = \mathbf{F}(0, \xi_j) \big(\mathbf{F}_\Theta'(\xi_j)\big)^{-1} = -\mathrm{i} \frac{\mu_0}{\omega^2} \Big[ \big(\mathbf{F}_\Theta^{\mathrm{a}}(-\xi_j)\big)^{\mathrm{T}} \Big]^{-1} \begin{pmatrix} q_P(\xi_j) & 0 \\ 0 & -q_S(\xi_j) \end{pmatrix} \big(\mathbf{F}_\Theta'(\xi_j)\big)^{-1} \tag{86}$$

*and* $\widehat{\mathbf{T}} = \widehat{\mathbf{T}}(\zeta)$, $\widehat{\mathbf{T}}(\zeta(\xi)) = \mathbf{T}(\xi)$ *with*

$$\mathbf{T}(\xi) = -\frac{\xi \mu_0}{\pi \omega^2} [(\mathbf{F}_\Theta^{\mathrm{a}})^{\mathrm{T}}(-\xi)]^{-1} \begin{pmatrix} q_P(\xi) & 0 \\ 0 & -q_S(\xi) \end{pmatrix} [\mathbf{F}_\Theta(\xi)]^{-1}, \quad \zeta \in \left(-\infty, \frac{\omega^2}{\mu_0}\right], \tag{87}$$

*signifying the branch cut.*

*Proof.* We fix a pole $\zeta_j = \xi_j^2$, use that

$$\mathbf{F}(x, \xi) [\mathbf{F}_\Theta(\xi)]^{-1} = \mathbf{S}(x, \xi) + \boldsymbol{\varphi}(x, \xi) \mathbf{M}(\xi)$$

[cf. (58) and (70)], and evaluate the residue

$$\mathbf{F}(x, \xi_j) \mathrm{Res}_{\xi=\xi_j} [\mathbf{F}_\Theta(\xi)]^{-1} = \boldsymbol{\varphi}(x, \xi_j) \mathrm{Res}_{\xi=\xi_j} \mathbf{M}(\xi).$$

As

$$\mathrm{Res}_{\zeta=\zeta_j} \widehat{\mathbf{M}}(\zeta) = 2\xi_j \mathrm{Res}_{\xi=\xi_j} \mathbf{M}(\xi) =: \alpha_j,$$

we get

$$\boldsymbol{\varphi}(x, \xi_j) \alpha_j = \mathbf{F}(x, \xi_j) u_j, \quad u_j = 2\xi_j \mathrm{Res}_{\xi=\xi_j} [\mathbf{F}_\Theta(\xi)]^{-1}. \tag{88}$$

In a similar fashion, we get

$$\boldsymbol{\varphi}^{\mathrm{a}}(x,\xi_j)\alpha_j = [u_j^{\mathrm{a}}]^{\mathrm{T}}[\mathbf{F}^{\mathrm{a}}(x,\xi_j)]^{\mathrm{T}}, \quad u_j^{\mathrm{a}} = 2\xi_j \mathrm{Res}_{\xi=\xi_j}[\mathbf{F}_\Theta^{\mathrm{a}}(\xi)]^{-1} \tag{89}$$

[cf. (64), (75), and (67)]. Integrating the derivative of the relevant Wronskian, we obtain

$$\lim_{x\to\infty} W(\mathbf{F}^{\mathrm{a}}(x,\xi_j),\mathbf{F}(x,\xi)) - W(\mathbf{F}^{\mathrm{a}}(x,\xi_j),\mathbf{F}(x,\xi))|_{x=0} = (\xi_j^2-\xi^2)\int_0^\infty [\mathbf{F}^{\mathrm{a}}(x,\xi_j)]^{\mathrm{T}}\mathbf{F}(x,\xi)\mathrm{d}x \tag{90}$$

for $\xi\in(\frac{\omega}{\sqrt{\mu_0}},\infty)$. Using the asymptotics of $\mathbf{F}_0(x,\xi)$ and $\mathbf{F}^{\mathrm{a}}(x,\xi_j)$ as $x\to\infty$, we get

$$\lim_{\xi\to\xi_j}\frac{1}{\xi_j^2-\xi^2}\lim_{x\to\infty} W(\mathbf{F}^{\mathrm{a}}(x,\xi_j),\mathbf{F}(x,\xi)) = 0. \tag{91}$$

Hence,

$$[u_j^{\mathrm{a}}]^{\mathrm{T}}\int_0^\infty [\mathbf{F}^{\mathrm{a}}(x,\xi_j)]^{\mathrm{T}}\mathbf{F}(x,\xi)\mathrm{d}x\, u_j = -\alpha_j, \tag{92}$$

yielding a representation of $\alpha_j$ in terms of the Jost solutions.

We now prove (86). As the pole $\zeta_j=\xi_j^2$ is simple, we also have

$$\alpha_j = \mathbf{F}(0,\xi_j)\left[\lim_{\xi\to\xi_j}(\xi^2-\xi_j^2)^{-1}\mathbf{F}_\Theta(\xi)\right]^{-1} = \frac{1}{2\xi_j}\mathbf{F}(0,\xi_j)[\mathbf{F}_\Theta'(\xi_j)]^{-1}. \tag{93}$$

At $\xi=\xi_j$, we have

$$\mathbf{F}'(0,\xi_j) = -\Theta(\xi_j)\mathbf{F}(0,\xi_j), \quad \mathbf{F}_\Theta^{\mathrm{a}}(\xi) = (\mathbf{F}^{\mathrm{a}})'(0,\xi)+\Theta^{\mathrm{T}}(\xi)\mathbf{F}^{\mathrm{a}}(0,\xi_j).$$

Then, using (57) at $\xi=\xi_j$, we get

$$\begin{aligned} W(\mathbf{F}^{\mathrm{a}}(0,-\xi_j)\mathbf{F}(0,\xi_j)) &= \left[((\mathbf{F}^{\mathrm{a}})')^{\mathrm{T}}(0,-\xi_j)+(\mathbf{F}^{\mathrm{a}})^{\mathrm{T}}(0,-\xi_j)\Theta(\xi_j)\right]\mathbf{F}(0,\xi_j) \\ &= (\mathbf{F}_\Theta^{\mathrm{a}}(-\xi_j))^{\mathrm{T}}\mathbf{F}(0,\xi_j) = -2\mathrm{i}\mu_0\frac{\xi_j}{\omega^2}\begin{pmatrix} q_P(\xi_j) & 0 \\ 0 & -q_S(\xi_j)\end{pmatrix}, \end{aligned}$$

and using (93), we obtain (86).

The jump across the branch cut is obtained through

$$\widehat{\mathbf{M}}^+(\zeta) = \widehat{\mathbf{M}}(\zeta+\mathrm{i}0) = \mathbf{M}(\xi^*) = \mathbf{F}(0,\xi^*)[\mathbf{F}_\Theta(\xi^*)]^{-1}, \quad \zeta\in\left(-\infty,\frac{\omega^2}{\mu_0}\right], \tag{94}$$

where $\xi^*\in[-\frac{\omega}{\sqrt{\mu_0}},\frac{\omega}{\sqrt{\mu_0}}]\cup\mathrm{i}\mathbb{R}$ approached from $\mathscr{K}_+$, and similarly,

$$\widehat{\mathbf{M}}^-(\zeta) = \widehat{\mathbf{M}}(\zeta-\mathrm{i}0) = \left([\mathbf{F}_\Theta^{\mathrm{a}}(-\xi^*)]^{\mathrm{T}}\right)^{-1}[\mathbf{F}^{\mathrm{a}}(0,-\xi^*)]^{\mathrm{T}}, \quad \zeta\in\left(-\infty,\frac{\omega^2}{\mu_0}\right]. \tag{95}$$

Using the definitions of the Jost functions and writing $\xi^*=\xi$, we then get

$$\widehat{\mathbf{T}}(\zeta(\xi)) = \frac{1}{2\pi\mathrm{i}}(\widehat{\mathbf{M}}^+(\zeta(\xi))-\widehat{\mathbf{M}}^-(\zeta(\xi))) = \frac{1}{2\pi\mathrm{i}}\left([\mathbf{F}_\Theta^{\mathrm{a}}]^{\mathrm{T}}(-\xi)\right)^{-1}W\left(\mathbf{F}^{\mathrm{a}}(x,-\xi),\mathbf{F}(x,\xi)\right)[\mathbf{F}_\Theta(\xi)]^{-1}. \tag{96}$$

With Lemma IV.1, we obtain (87). □

In (83), we distinguish, from a physics perspective, the following three contributions:

$$\int_{-\infty}^0 \frac{\widehat{\mathbf{T}}(\eta)}{\zeta-\eta}\mathrm{d}\eta$$

from the evanescent modes,

$$\int_0^{\frac{\omega^2}{\mu_0}} \frac{\widehat{\mathbf{T}}(\eta)}{\zeta-\eta}\mathrm{d}\eta$$

from the radiating modes, and

$$\sum_{j=1}^{N} \frac{\alpha_j}{\zeta - \zeta_j}$$

from the guided modes.

*Remark V.2. We note that through (86), (87) and using (55), $\alpha_j$ and* $\mathbf{T}$ *can be expressed in terms of the Jost function only. Thus, Lemma V.2 indicates that the Jost function encodes the boundary spectral data.*

## B. Even extension

In the original variable $\xi$, $\xi^2 = \zeta$, (83) reads

$$\begin{aligned}\mathbf{M}(\xi) &= \int_{-\infty}^{\frac{\omega^2}{\mu_0}} \frac{\widehat{\mathbf{T}}(\eta)}{\xi^2 - \eta} \mathrm{d}\eta + \sum_{j=1}^{N} \frac{\alpha_j}{\xi^2 - \xi_j^2} \\ &= \int_{-\infty}^{\frac{\omega^2}{\mu_0}} \frac{\widehat{\mathbf{T}}(\eta)}{\xi^2 - \eta} \mathrm{d}\eta + \sum_{j=1}^{N} \frac{\alpha_j}{2\xi_j} \frac{1}{\xi - \xi_j} + \sum_{j=1}^{N} \frac{\alpha_j}{2\xi_j} \frac{(-1)}{\xi + \xi_j}, \qquad \xi \in \mathscr{K}_+ \backslash \{\xi_j\}_{j=1}^{N}.\end{aligned} \tag{97}$$

Using this representation, $M(\xi)$ has an artificial extension to

$$\mathbb{C} \backslash \bigcup \left\{ \mathrm{i}\mathbb{R}, \left[ -\frac{\omega}{\sqrt{\mu_0}}, \frac{\omega}{\sqrt{\mu_0}} \right], \{\pm \xi_j\}_{j=1}^{N} \right\} \equiv \mathscr{K} \backslash \{\pm \xi_j\}_{j=1}^{N}$$

as an even function

$$\mathbf{M}(-\xi) = \mathbf{M}(\xi),$$

which we will employ in the further analysis. We emphasize that this extension is fundamentally different from the above-mentioned meromorphic continuation.

In the further analysis, we invoke the following.

*Assumption V.2. The parameter functions, $\lambda$ and $\mu$, are such that there is no pole of* $\mathbf{M}(\xi)$ *with* $\operatorname{Im} q_S = 0$ *except, possibly, at* $\xi = \frac{\omega}{\sqrt{\mu_0}}$ *as a one-sided limit in* $\mathscr{K}_+$.

## C. Data for the original Rayleigh system

### 1. Weyl matrix

In Appendix C, we develop a relation between the Neumann-to-Dirichlet map ($\mathbf{ND}$) of the original Rayleigh system and the Weyl matrix induced by the Markushevich substitutions. From

$$\mathbf{ND}(\xi) = \left[ \begin{pmatrix} \mathrm{i}\dfrac{\mu_0}{\mu(0)} & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \dfrac{1}{2}\mathrm{i} \\ -\dfrac{\mu_0}{\mu(0)}\xi & 0 \end{pmatrix} \mathbf{M}(\xi) \right] \begin{pmatrix} \dfrac{1}{2\mu_0 \xi} & 0 \\ \dfrac{\mu'(0)}{\mu^2(0)} \dfrac{1}{\xi} & \dfrac{\mathrm{i}}{\mu(0)} \end{pmatrix}$$

[cf. (C13)], it follows that $\mathbf{ND}$ and $\mathbf{M}$ have the same poles and their jumps across branch cuts are explicitly related; this relationship depends on $\mu(0)$, $\mu'(0)$, and $\mu_0$.

### 2. Jost solution

In addition to the Weyl matrix, we need the Jost solution at $x = 0$ as the data. We let $\mathbf{w} = [w_P \; w_S]$ denote the Jost solution of the Rayleigh system before the Markushevich transform, that is, both columns of $\mathbf{w}$ satisfy (7) and (8) and conditions

$$\mathbf{w} = [w_{P,0}^{+} \; w_{S,0}^{+}] \qquad \text{for} \quad x \geq H,$$

where the "reference" Jost solution comprised of $w_{P,0}+$ and $w_{S,0}^{+}$ is given in (44) and (45). The Jost solution can be excited by imposing the "outgoing radiation" conditions at the bottom of the slab,

$$\mathbf{w}'(H, \cdot) - \mathrm{i}\, \mathbf{w}(H, \cdot) \begin{pmatrix} q_P & 0 \\ 0 & q_S \end{pmatrix} = 0, \tag{98}$$

supplemented with a Dirichlet boundary value

$$\mathbf{w}(H,\xi) = [w_P(H,\xi)\ w_S(H,\xi)] = [w^+_{P,0}(H,\xi)\ w^+_{S,0}(H,\xi)]$$

at the bottom of the slab, $x = H$, and observed at $x = 0$, giving $\mathbf{w}(0,\xi)$. Upon the Markushevich substitution, this yields $\mathbf{F}(0,\xi)$.

#### 3. Jost function

The Jost functions $\mathbf{F}_\Theta(\xi)$ and $\mathbf{F}^{\mathrm{a}}_\Theta(\xi)$ can be considered as alternative data. By (C4) and (C5), these are directly related to the boundary matrix of the original Rayleigh problem [cf. (C3)],

$$\mathbf{B}(\mathbf{w}) = \begin{pmatrix} b_-(w_P) & b_-(w_S) \\ a_-(w_P) & a_-(w_S) \end{pmatrix},$$

where

$$\mathbf{w} = [w_P\ w_S] \tag{99}$$

is the Jost solution discussed above. By (55), the Jost function determines the adjoint Jost function if $\mu(0)$, $\mu'(0)$, and $\mu_0$ are known.

### D. Unique recovery of a potential of Lamé type

In the following lemma, proposition, and theorem, we assume that $H$, $\lambda_0$, $\mu_0$, $\mu(0)$, and $\mu'(0)$ are known. We introduce the expansion of the Jost solution at the boundary,

$$\mathbf{F}(0,\xi) = \xi\mathbf{G}_0(0,\xi) + \mathbf{G}_1(0) + \mathbf{R}(\xi), \quad \mathbf{R}(\xi) = \mathscr{O}\left(\frac{1}{|\xi|}\right). \tag{100}$$

In Theorem VI.1, we will construct explicit expressions for $\mathbf{G}_0(0,\xi)$ and $\mathbf{G}_1(0,\xi)$.

*Lemma V.3. Given $\lambda_0$ and $\mu_0$. The mapping from $G^H$ [cf. (30)] to $[\mathbf{G}_0(0,\xi), \mathbf{G}_1(0,\xi)]$ for any pair of frequencies, $\omega_1 \neq \omega_2 \in \mathbb{R}_+$, is an injection.*

The proof is given in Subsection VII B. Thus, $[\mathbf{G}_0(0,\xi), \mathbf{G}_1(0,\xi)]$ for any two frequencies $\omega_1 \neq \omega_2 \in \mathbb{R}_+$ determine $G^H$. Moreover, $G^H$ together with $H$, $\lambda_0$, $\mu_0$, and $\omega$ determine $Q_0$.

*Proposition V.1. Given $G^H$. For $\omega$ fixed, let $V_1, V_2$ be compactly supported on $[0,H]$ and belong to $L^1([0,H])$ with associated Weyl matrices $\mathbf{M}_1$, $\mathbf{M}_2$. If $H$, $\lambda_0$, $\mu_0$, $\mu(0)$, and $\mu'(0)$ are known and Assumptions V.1 and V.2 hold true, then $\mathbf{M}_2(\xi) = \mathbf{M}_1(\xi)$ for all $\xi \in \mathscr{K}_+$ implies that $V_2 = V_1$.*

The proof is given in Subsection VII A. Thus, $G^H$ together with $\mathbf{M}(\xi)$ determine $V$. By implication, $[\mathbf{G}_0(0,\xi), \mathbf{G}_1(0,\xi)]$ for any two frequencies $\omega_1 \neq \omega_2 \in \mathbb{R}_+$ and $\mathbf{M}(\xi)$ determine $Q$. Furthermore, from a Lamé-type $Q$ for any pair of frequencies, $\omega_1 \neq \omega_2 \in \mathbb{R}_+$, we recover $\lambda$ and $\mu$, which is proved in Subsection VII C.

We need both the Weyl function, or **ND** map, and the Jost solution at the boundary for the unique recovery of Lamé parameters. Alternatively, we may use the Jost function $\mathbf{F}_\Theta(\xi)$ as the data, as by Lemma V.2 and Remark V.2, assuming that $\lambda_0$ and $\mu_0$ are known, $\mathbf{F}_\Theta(\xi)$ determines the Weyl function $\mathbf{M}$ and, by (59), $\mathbf{F}(0,\xi)$. We recall that $\mathbf{F}_\Theta(\xi)$ also determines $\mathbf{F}^{\mathrm{a}}_\Theta(\xi)$. Thus, from the analysis above, replacing the data in Proposition V.1, we obtain the following.

**Theorem V.1.** *Let $Q_1$, $Q_2$ be of Lamé type with the associated Jost functions $\mathbf{F}_{\Theta;1}$, $\mathbf{F}_{\Theta;2}$. Assume that $H$, $\lambda_0$, $\mu_0$, $\mu(0)$, and $\mu'(0)$ are known. Then, $\mathbf{F}_{\Theta;2}(\xi) = \mathbf{F}_{\Theta;1}(\xi)$ for all $\xi \in \mathscr{K}_+$ and any pair of frequencies, $\omega_1 \neq \omega_2 \in \mathbb{R}_+$, subject to Assumptions V.1 and V.2, implies that $Q_2 = Q_1$.*

## VI. ASYMPTOTIC EXPANSIONS

### A. Jost solutions

In this subsection, we establish fundamental properties of the Jost solutions. We refer to Appendix B for a representation of the Green's function $\mathfrak{G} = \mathfrak{G}(x,y)$ introduced in (71).

**Theorem VI.1.** *For any fixed $x \geq 0$, the Jost solution, $\mathbf{F}$, is analytic in $\xi$ on $\mathscr{K}_+$, of exponential type, and satisfies*

$$\mathbf{F}(x,\xi) = \mathbf{F}_0(x,\xi) \overbrace{- \int_x^H \mathfrak{G}(x,y)V(y)\mathbf{F}_0(y,\xi)\mathrm{d}y}^{\mathbf{F}_1(x,\xi)} + \sum_{k=2}^{\infty} \mathbf{F}_k(x,\xi),$$

$$\mathbf{F}_k(x,\xi) = \frac{|\xi|}{k!}\mathscr{O}\left(\frac{1}{\max\{|\xi|,1\}}\right)^k e^{-x\,\mathrm{Im}\,q_S(\xi)}.$$

*As* $|\xi|\to\infty$, $\xi\in\mathscr{K}_+$,

$$\mathbf{F}(x,\xi) = e^{-x\xi}\xi\left(\mathbf{G}_0(x,\xi) + \frac{1}{\xi}\mathbf{G}_1(x) + \mathscr{O}\left(\frac{1}{|\xi|^2}\right)\right), \tag{101}$$

*where*

$$\mathbf{G}_0(x,\xi) = -\frac{\mu_0}{\omega^2}\begin{pmatrix} G_{11}^H & G_{11}^H \\ G_{12}^H & G_{12}^H \end{pmatrix} + \frac{1}{\xi}\begin{pmatrix} \frac{c_0}{2}G_{11}^H H + G_{21}^H & -\frac{1}{2}xG_{11}^H \\ \frac{c_0}{2}G_{12}^H H + G_{22}^H & -\frac{1}{2}xG_{12}^H \end{pmatrix} \tag{102}$$

*and*

$$\mathbf{G}_1(x) = -\frac{1}{2}\frac{\mu_0}{\omega^2}\int_x^H V(y)\mathrm{d}y\begin{pmatrix} G_{11}^H & G_{11}^H \\ G_{12}^H & G_{12}^H \end{pmatrix}. \tag{103}$$

*Analogous properties and an expansion can be obtained for* $\mathbf{F}^{\mathrm{a}}$.

*Remark VI.1. The Proof of Theorem VI.1 is based on an iteration of Volterra-type equation (72) [and (77)] following a standard argument.*[10] *We note that the very reason to perform the Markushevich transform was to re-write the Rayleigh problem in the Schrödinger form and then as a Volterra-type integral equation with a bounded kernel.*

In the above theorem, $\mathbf{G}_0$ *contains a contribution* $\mathscr{O}(\frac{1}{|\xi|})$*; this contribution is essential to ensure that* $\mathbf{G}_0(0,\xi)$ *is invertible while* $\mathbf{G}_0$ *only depends on* $Q_0$. We write

$$\mathbf{F}(0,\xi) = \xi\mathbf{G}_0(0,\xi) + \mathbf{G}_1(0) + \mathbf{R}(\xi), \quad \mathbf{R}(\xi) = \mathscr{O}\left(\frac{1}{|\xi|}\right). \tag{104}$$

Furthermore, the Jost function, $\mathbf{F}_\Theta$, and $\det\mathbf{F}_\Theta$ are analytic in $\xi$ where $\mathrm{Im}\,q_P > 0, \mathrm{Im}\,q_S > 0$ ($\xi\in\mathscr{K}_+$; see Appendix A) and continuous in $\xi$ where $\mathrm{Im}\,q_P \geq 0, q_P \neq 0, \mathrm{Im}\,q_S \geq 0, q_S \neq 0$. We obtain the following.

*Corollary VI.1. The Jost function admits the asymptotic expansion, as* $|\xi|\to\infty$, $\xi\in\mathscr{K}_+$,

$$\begin{aligned}\mathbf{F}_\Theta(\xi) = &-2\xi^3\frac{\mu_0^2}{\omega^2\mu(0)}G_{11}^H\begin{pmatrix}0 & 0\\ 1 & 1\end{pmatrix} + \xi^2\left(\frac{\mu_0}{\omega^2}\begin{pmatrix} G_{11}^H & G_{11}^H \\ G_{12}^H & G_{12}^H \end{pmatrix}\right.\\ &+ 2\frac{\mu_0}{\mu(0)}\left(G_{11}^H\frac{1}{2}c_0H + G_{21}^H\right)\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\\ &\left.-\frac{1}{2}\frac{\mu_0}{\omega^2}\begin{pmatrix}0 & 0\\ 2\frac{\mu_0}{\mu(0)} & 0\end{pmatrix}\int_0^H V(y)\mathrm{d}y\begin{pmatrix} G_{11}^H & G_{11}^H \\ G_{12}^H & G_{12}^H \end{pmatrix}\right) + o(|\xi|^2).\end{aligned} \tag{105}$$

*Corollary VI.2. Solutions* $\mathbf{S}(x,\xi)$ *and* $\boldsymbol{\varphi}(x,\xi)$ *[cf. (70) and above] are entire on* $\mathbb{C}$ *and even functions of* $\xi$ *of exponential type. For the first and second derivatives, the following estimate holds true:*

$$\|\boldsymbol{\varphi}^{(k)}(x,\xi)\| \leq C|\xi|^{k+1}e^{|\mathrm{Re}\,\xi|x}, \quad \xi\in\mathscr{K}_+, \quad k=0,1, \tag{106}$$

*uniformly in* $x\geq 0$.

## B. Weyl matrix

Next, we study the Weyl matrix introduced in (59) and derive its asymptotic expansion as $|\xi|\to\infty$, $\xi\in\mathscr{K}_+$. It appears that it is more convenient to start the analysis with its inverse,

$$\mathbf{M}^{-1}(\xi) = \mathbf{F}'(0,\xi)\mathbf{F}^{-1}(0,\xi) + \Theta(\xi), \tag{107}$$

where $\Theta$ is given in (19). The following result is an analog of Ref. 8, Proposition 2.

*Lemma VI.1. The Weyl matrix,* $\mathbf{M}(\xi)$*, and its inverse* $\mathbf{M}(\xi)^{-1}$ *have the following asymptotic expansions for* $|\xi| \to \infty$, $\xi \in \mathscr{K}_+$*:*

1. *If* $Q \in C^\infty$ *and all its derivatives are integrable on* $\mathbb{R}_+$*, then* $\mathbf{M}(\xi)^{-1}$ *has an asymptotic expansion to all orders,*

$$\mathbf{M}(\xi)^{-1} = \Theta(\xi) + \xi \sum_{k=0}^{\infty} \xi^{-k} X_k(0) \tag{108}$$

$$X_0 = -I_2, \quad X_1 = 0, \quad X_2 = -\frac{1}{2}Q, \quad X_3 = -\frac{1}{4}Q', \quad 2X_{k+1} = X_k' + \sum_{j=1}^{k} X_j X_{4-j}$$

*[cf. (19)].*

2. *If* $Q \in C^\infty$ *and all its derivatives are integrable on* $\mathbb{R}_+$*, then* $\mathbf{M}(\xi)$ *has an asymptotic expansion to all orders,* $\mathbf{M}(\xi) = \sum_{k=0}^{\infty} \xi^{-k} Y_k$*. Weakening the condition, if* $Q \in C^3$ *and its first three derivatives are integrable on* $\mathbb{R}_+$*,*

$$\mathbf{M}(\xi) = Y_0 + \xi^{-1} Y_1 + \xi^{-2} Y_2 + \xi^{-3}(Y_3 + \boldsymbol{\alpha}(\xi)), \tag{109}$$

*where* $\boldsymbol{\alpha}(\xi) \to 0$ *as* $|\xi| \to \infty$, $\xi \in \mathscr{K}_+$*, with*

$$Y_0 = \frac{1}{1 - 2\varpi\theta_2} \begin{pmatrix} 0 & 0 \\ y_{0;21} & 0 \end{pmatrix}, \tag{110}$$

$$Y_1 = \frac{1}{1 - 2\varpi\theta_2} \begin{pmatrix} -1 & 0 \\ y_{1;21} & -1 \end{pmatrix}, \tag{111}$$

$$Y_2 = \frac{1}{1 - 2\varpi\theta_2} \begin{pmatrix} y_{2;11} & y_{2;12} \\ y_{2;21} & y_{2;22} \end{pmatrix}, \tag{112}$$

*in which [cf. (19)]* $1 - 2\varpi\theta_2 = \frac{\lambda(0)+\mu(0)}{\lambda(0)+2\ \mu(0)}$ *and*

$$\begin{aligned}
y_{0;21} &= -2\varpi, \\
y_{1;21} &= \frac{(1-\varpi)Q_{12}(0)}{(1-2\varpi\theta_2)\theta_2} + \frac{2\varpi(\theta_3 + \varpi Q_{12}(0))}{1 - 2\varpi\theta_2}, \\
y_{2;11} &= \frac{1}{2} y_{1;21}, \\
y_{2;12} &= -\theta_2, \\
y_{2;21} &= \frac{Q_{12}'(0) - y_{1;21} Q_{12}(0) + Q_{11}(0) + Q_{22}(0) - y_{1;21}\theta_3 + \theta_1}{1 - 2\theta_2}, \\
y_{2;22} &= \frac{\varpi Q_{12}(0) + 2\varpi\theta_2\theta_3}{1 - 2\varpi\theta_2}.
\end{aligned}$$

*Proof.* We first prove (a). Where $\mathbf{F}(x,\xi)$ is invertible ($\xi \neq 0$), (68) is equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mathbf{F}'(x,\xi)\mathbf{F}(x,\xi)^{-1}) + (\mathbf{F}'(x,\xi)\mathbf{F}(x,\xi)^{-1})^2 = Q(x) + \xi^2. \tag{113}$$

Using analytic properties of the Jost solution, it follows that $\mathbf{F}_0(x,\xi)^{-1}\mathbf{F}(x,\xi)$ [cf. (52)] admits an expansion to all orders of $\xi^{-1}$ and that this expansion can be differentiated term by term. Such an expansion also exists for $\mathbf{F}(x,\xi)$ and, hence, for $\mathbf{F}'(x,\xi)$. Thus, the following expansion exists:

$$\mathbf{F}'(x,\xi)\mathbf{F}(x,\xi)^{-1} = \xi \sum_{k=0}^{\infty} \xi^{-k} X_k(x).$$

We insert this expansion into (113) and note that

$$\mathbf{M}(\xi)^{-1} = \Theta(\xi) + \mathbf{F}'(x,\xi)\mathbf{F}(x,\xi)^{-1}|_{x=0}.$$

We prove (b) using (a) and $\mathbf{M}(\xi)\mathbf{M}(\xi)^{-1} = I_2$ by explicit calculations. □

Capturing the leading orders, we introduce

$$\widetilde{Y}(\xi) = Y_0 + \frac{1}{\xi} Y_1 = \frac{1}{\xi(1-2\varpi\theta_2)} \begin{pmatrix} -1 & 0 \\ 2(b-\varpi\xi) & -1 \end{pmatrix}, \quad b = \frac{1}{2} y_{1;21}, \tag{114}$$

with inverse

$$\widetilde{Y}(\xi)^{-1} = \xi^2(1-2\varpi\theta_2) \begin{pmatrix} 0 & 0 \\ 2\varpi & 0 \end{pmatrix} + \xi(1-2\varpi\theta_2) \begin{pmatrix} -1 & 0 \\ -2b & -1 \end{pmatrix}$$

and the properties

$$Y_0 \widetilde{Y}(\xi)^{-1} = \xi \begin{pmatrix} 0 & 0 \\ 2\varpi & 0 \end{pmatrix}, \tag{115}$$

$$Y_1 \widetilde{Y}(\xi)^{-1} = \widetilde{Y}(\xi)^{-1} Y_1 = \xi \begin{pmatrix} 1 & 0 \\ -2\varpi\xi & 1 \end{pmatrix} = \xi E(-\xi), \tag{116}$$

where

$$E(\xi) = \begin{pmatrix} 1 & 0 \\ 2\varpi\xi & 1 \end{pmatrix} \tag{117}$$

forms a group as $E(\xi)^{-1} = E(-\xi)$ and

$$E(\xi_1)E(\xi_2)^{-1} = \begin{pmatrix} 1 & 0 \\ 2\varpi(\xi_1 - \xi_2) & 1 \end{pmatrix}.$$

Then,

$$\mathbf{M}(\xi)\widetilde{Y}(\xi)^{-1} = T_0 - \frac{1}{\xi} T_1 + \mathscr{O}\left(\frac{1}{|\xi|^2}\right), \quad \xi \in \mathscr{K}_+, \tag{118}$$

where

$$T_0 = \begin{pmatrix} 1 - 2\varpi\theta_1 & 0 \\ 2\varpi(b-\theta_3) & 1 \end{pmatrix}. \tag{119}$$

From (109) and (96) with (94) and (95), it follows that

$$\mathbf{T}(\zeta) = \frac{1}{\pi \mathrm{i} \xi}\left( Y_1 + \frac{1}{\xi^2} Y_3 + o\left(\frac{1}{|\xi|^2}\right)\right), \quad \zeta \in \left(-\infty, \frac{\omega^2}{\mu_0}\right] \tag{120}$$

[cf. (87)], where $\xi = \sqrt{\zeta}$ is defined below (82), while on the branch cut, if $\zeta < 0$, then $\xi \in \mathrm{i}\mathbb{R}$.

## VII. GEL'FAND–LEVITAN TYPE EQUATION AND PROOF OF THEOREM V.1

Using the results from Sec. VI, we obtain an asymptotic expansion for $\mathbf{F}(x,\xi)[\mathbf{F}(0,\xi)]^{-1}$ in the following.

*Lemma VII.1. The following asymptotic expansion holds true:*

$$\mathbf{F}(x,\xi)[\mathbf{F}(0,\xi)]^{-1} = e^{-x\xi}\left(I_2 + \frac{1}{\xi}\mathbf{D}(x) + o\left(\frac{1}{|\xi|}\right)\right), \quad \xi \in \mathscr{K}_+,$$

*where*

$$\mathbf{D}(x) = \frac{1}{2}\int_0^x V(y)\mathrm{d}y \begin{pmatrix} -G_{11}^H\left(\frac{c_0}{2}G_{12}^H H + G_{22}^H\right) & G_{11}^H\left(\frac{c_0}{2}G_{11}^H H + G_{21}^H\right) \\ -G_{12}^H\left(\frac{c_0}{2}G_{12}^H H + G_{22}^H\right) & G_{12}^H\left(\frac{c_0}{2}G_{11}^H H + G_{21}^H\right) \end{pmatrix}.$$

*Proof.* The statement follows from an explicit calculation using Theorem VI.1, that is, (101)–(103) and (100). □

From (60) upon employing Lemma VI.1, that is, (109), and using

$$\mathbf{\Phi}(x,\pm\xi) = \mathbf{F}(x,\pm\xi)[\mathbf{F}(0,\pm\xi)]^{-1}\mathbf{M}(\pm\xi), \quad \pm\xi \in \mathscr{K}_+ \tag{121}$$

(and extension to the branch cuts), we obtain the following.

*Corollary VII.1. The following asymptotic expansion holds true:*

$$\mathbf{\Phi}(x,\pm\xi) = e^{-\xi x}\left(Y_0 \pm \frac{1}{\xi}\left(Y_1 + \left(\mathbf{D}(x) \mp \frac{\omega^2}{2\mu_0}x\right)Y_0\right) + o\left(\frac{1}{|\xi|}\right)\right), \quad \pm\xi \in \mathscr{K}_+.$$

This corollary immediately implies that

$$\mathbf{\Phi}'(x,\pm\xi) = -\xi e^{-\xi x}\left(Y_0 \pm \frac{1}{\xi}\left(Y_1 + \left(\mathbf{D}(x) \mp \frac{\omega^2}{2\mu_0}x\right)Y_0\right) + o\left(\frac{1}{|\xi|}\right)\right), \quad \pm\xi \in \mathscr{K}_+. \tag{122}$$

### A. Recovery of $V$ assuming that $G^H$ is known

For the Proof of Theorem V.1, we change variables through the transformation (Appendix A)

$$\xi \to k, \quad k = \sqrt{\frac{\omega^2}{\mu_0} - \xi^2}, \quad \xi = -\mathrm{i}k + \mathscr{O}\left(\frac{1}{|k|}\right), \quad \xi \in \mathscr{K} \quad (\omega \text{ fixed}). \tag{123}$$

The choice of sign is determined by letting $\xi \in \mathscr{K}_+$ [where also $\operatorname{Im} q_P(\xi) > 0$] correspond to $k \in \mathbb{C}_+$ $(\operatorname{Im} k > 0)$. The inverse of the transformation is defined on $\mathbb{C}_+$ and written as $\xi(k)$. The branch cut in $\xi \in \mathscr{K}_S$ corresponds with $\operatorname{Im} k = 0$. We let

$$\mathscr{K}_+ \to \mathscr{K}_- : \ \xi \to -\xi \quad \text{correspond with} \quad \mathbb{C}_+ \to \mathbb{C}_- : \ k \to -k,$$

where $\mathbb{C}_+$ $(\operatorname{Im} k < 0)$. First, we give some basic asymptotic expansions. To next order, we have

$$\mathrm{i}k + \xi = \frac{\omega^2}{2\xi\mu_0} + \mathscr{O}\left(\frac{1}{|\xi|^2}\right), \qquad \xi \in \mathscr{K}_S,$$

so that

$$e^{\mp(\mathrm{i}k+\xi)x} = 1 \mp \frac{\omega^2}{2\xi\mu_0}x + \mathscr{O}\left(\frac{1}{|\xi|^2}\right), \quad e^{\mp\mathrm{i}kx} = e^{\pm\xi x}\left(1 \mp \frac{\omega^2}{2\xi\mu_0}x + \mathscr{O}\left(\frac{1}{|\xi|^2}\right)\right) \qquad \text{as } |\xi| \to \infty.$$

We introduce

$$\mathbf{M}_\pm(k) = \mathbf{M}(\pm\xi), \quad \pm\xi \in \mathscr{K}_+, \quad \pm\operatorname{Im} k \geq 0, \tag{124}$$

and we introduce two more solutions [cf. (70)]

$$\mathbf{\Phi}_\pm(x,k) = \mathbf{\Phi}(x,\pm\xi) = \widetilde{\mathbf{S}}(x,k) + \widetilde{\boldsymbol{\varphi}}(x,k)\mathbf{M}_\pm(k), \quad \pm\xi \in \mathscr{K}_+, \quad \pm\operatorname{Im} k \geq 0, \tag{125}$$

where we identify

$$\widetilde{\mathbf{S}}(x,k) = \mathbf{S}(x,\xi(k)), \qquad \widetilde{\boldsymbol{\varphi}}(x,k) = \boldsymbol{\varphi}(x,\xi(k)) \tag{126}$$

with

$$\widetilde{\Theta}(k) = \Theta(\xi(k)). \tag{127}$$

$\widetilde{\mathbf{S}}(x,k)$, $\widetilde{\boldsymbol{\varphi}}(x,k)$, and $\widetilde{\Theta}(k)$ are entire and even in $k$ as solutions and by the relevant boundary conditions. While $\mathbf{M}_+$ is defined for $k \in \mathbb{C}_+$ $(\operatorname{Im} k > 0)$, $\mathbf{M}_-$ is defined for $k \in \mathbb{C}_-$ $(\operatorname{Im} k < 0)$. It follows that for *real-valued* $k$,

$$\mathbf{\Phi}_+(x,k) - \mathbf{\Phi}_-(x,k) = \widetilde{\boldsymbol{\varphi}}(x,k)(\mathbf{M}_+(k) - \mathbf{M}_-(k)). \tag{128}$$

Applying Lemma VI.1 and using (114), we find that

$$\frac{2}{\xi(k)}Y_1 = \widetilde{Y}(\xi(k)) - \widetilde{Y}(-\xi(k)) = \mathbf{M}(\xi(k)) - \mathbf{M}(-\xi(k)) + \mathscr{O}\left(\frac{1}{|k|^3}\right). \tag{129}$$

In the later analysis, for real-valued $k$, we will employ the notation $\widetilde{Y}(-\mathrm{i}k)$ and $E(-\mathrm{i}k)$, substituting $-\mathrm{i}k$ for $\xi$ [or $\xi(k)$]. By abuse of notation, in this subsection, we will omit $\,\widetilde{}\,$ in $\widetilde{\mathbf{S}}$, $\widetilde{\boldsymbol{\varphi}}$, and $\widetilde{\Theta}$.

Clearly, the matrix functions $\mathbf{\Phi}_\pm$ satisfy the following boundary conditions [cf. (63)]:

$$\mathbf{\Phi}_\pm(0,k) = \mathbf{M}_\pm(k), \quad \mathbf{\Phi}'_\pm(0,k) + \Theta(k)\mathbf{\Phi}_\pm(0,k) = I_2, \quad \pm\mathrm{Im}\,k > 0.$$

Using Corollary VII.1 and (128), for real-valued $k$, we note that

$$\mathbf{\Phi}_+(x,k) - \mathbf{\Phi}_-(x,k) = \boldsymbol{\varphi}(x,k)\left(\frac{2}{-\mathrm{i}k}Y_1 + \mathscr{O}\left(\frac{1}{|k|^3}\right)\right), \tag{130}$$

and the expansions in Corollary VII.1 imply the following.

*Lemma VII.2. For $k \in \mathbb{R}$, the following asymptotic expansion holds true:*

$$\boldsymbol{\varphi}(x,k) = \boldsymbol{\phi}_0(x,k) + (e^{\mathrm{i}kx} + e^{-\mathrm{i}kx})\mathbf{m_1}(x) + (e^{\mathrm{i}kx} - e^{-\mathrm{i}kx})\mathscr{O}\left(\frac{1}{|k|}\right), \tag{131}$$

*where*

$$\boldsymbol{\phi}_0(x,k) = -\frac{1}{2}\mathrm{i}k(e^{\mathrm{i}kx} - e^{-\mathrm{i}kx})Y_0Y_1^{-1}, \tag{132}$$

*in which*

$$\frac{1}{2}Y_0Y_1^{-1} = \varpi(1-2\varpi\theta_2)\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}$$

*and $\mathbf{m_1}(x)$ is independent of $k$.*

The lemma above implies that for $k \in \mathbb{R}$,

$$\boldsymbol{\varphi}(x,k)\mathbf{M}_\pm(k) = \mathscr{O}(1),$$

and we have

$$e^{-|k|x}\boldsymbol{\varphi}(x,k) = \mathrm{i}k\frac{1}{2}Y_0Y_1^{-1} + \mathscr{O}(1). \tag{133}$$

We introduce

$$\widehat{\boldsymbol{\varphi}}(x,k) = \boldsymbol{\varphi}(x,k) - \boldsymbol{\phi}_0(x,k) \tag{134}$$

[cf. (132)] for the later analysis.

*Definition VII.1. We let*

$$\Psi(x,k) = \begin{cases} \Psi_+(x,k) = e^{\mathrm{i}kx}\left(\mathbf{\Phi}_+(x,k) - \dfrac{2\mathrm{i}}{k}\boldsymbol{\varphi}(x,k)Y_1\right)\widetilde{Y}^{-1}(\mathrm{i}k), & k \in \mathbb{C}_+,\\ \Psi_-(x,k) = e^{\mathrm{i}kx}\mathbf{\Phi}_-(x,k)\widetilde{Y}^{-1}(\mathrm{i}k), & k \in \mathbb{C}_-.\end{cases}$$

*[cf. (114)].*

The matrix function $\Psi$ is meromorphic and defined on $\mathbb{C}$ through the above-mentioned extension of $\mathbf{\Phi}$ to branch cuts, which is important for the later contour integration in the Proof of Proposition VII.2. However, as elucidated in Subsection V B, throughout we intrinsically use functions defined on the physical sheet only.

From (121), it follows that $\Psi$ inherits its poles from $\mathbf{M}$ (see Subsection V A).

*Lemma VII.3. The matrix function $\Psi$ has poles at $\pm k_j$ with*

$$k_j = \sqrt{\frac{\omega^2}{\mu_0} - \xi_j^2}.$$

*The residues are given by*

$$\mathrm{Res}_{k=\pm k_j}\Psi(x,k) = \frac{1}{2\mathrm{i}}e^{\pm\mathrm{i}k_jx}\boldsymbol{\varphi}(x,k_j)C_jE(\mp\mathrm{i}k_j), \quad C_j = -2k_jB_jY_1^{-1} = \alpha_jY_1^{-1}. \tag{135}$$

*Proof.* We have [cf. (125)]

$$\Psi(x,k) = e^{\mathrm{i}kx}\left(\mathbf{S}(x,k)Y_1^{-1} + \boldsymbol{\varphi}(x,k)\mathbf{M}_+(k)Y_1^{-1} - \frac{2\mathrm{i}}{k}\boldsymbol{\varphi}(x,k)\right)Y_1\widetilde{Y}(\mathrm{i}k)^{-1}$$

for $\operatorname{Im} k > 0$, where $\mathbf{S}$ and $\boldsymbol{\varphi}$ are entire in $k$. As

$$\begin{aligned}\alpha_j &= \lim_{\xi\to\xi_j}(\xi^2 - \xi_j^2)\mathbf{M}(\xi) = -\lim_{k\to k_j}(k^2 - k_j^2)\mathbf{M}_+(k)\\ &= -2k_j\lim_{k\to k_j}(k-k_j)\mathbf{M}_+(k) = -2k_jB_j, \quad B_j = \operatorname{Res}_{k=k_j}\mathbf{M}_+(k),\end{aligned} \tag{136}$$

we obtain

$$\operatorname{Res}_{k=k_j}\Psi(x,k) = e^{\mathrm{i}k_jx}\boldsymbol{\varphi}(x,k_j)B_jY_1^{-1}Y_1\widetilde{Y}(\mathrm{i}k_j)^{-1},$$

which, with (116), implies the statement. Here, we use that the poles of $\mathbf{M}$ are simple. □

We define three new matrix functions in the following.

*Definition VII.2. We let $j(k)$, $e(x,k)$, and $\tilde{e}(x,k)$ be given by*

$$\mathbf{M}_+(k) - \widetilde{Y}(-\mathrm{i}k) = -j(k)Y_1, \quad k\in\mathbb{C}_+, \tag{137}$$

$$\tilde{e}(x,k) = \frac{e^{\mathrm{i}kx}}{2\mathrm{i}k}E(-\mathrm{i}k) - \frac{e^{-\mathrm{i}kx}}{2\mathrm{i}k}E(\mathrm{i}k) = \frac{1}{k}\begin{pmatrix}\sin kx & 0\\ -2\varpi k\cos kx & \sin kx\end{pmatrix}, \quad k\in\mathbb{C}, \tag{138}$$

$$e(x,k) = \tilde{e}'(x,k) = \frac{1}{2}e^{\mathrm{i}kx}E(-\mathrm{i}k) + \frac{1}{2}e^{-\mathrm{i}kx}E(\mathrm{i}k) = \begin{pmatrix}\cos kx & 0\\ 2\varpi k\sin kx & \cos kx\end{pmatrix}, \quad k\in\mathbb{C}. \tag{139}$$

Through the definition of $\mathbf{M}_-$, we obtain the extension of $j(k)$ from $\mathbb{C}_+$ to $\mathbb{C}_-$. It follows that

$$j(k) = \begin{cases} j(k), & k\in\mathbb{C}_+,\\ j(-k), & k\in\mathbb{C}_-,\end{cases} \tag{140}$$

that is, for $\operatorname{Im} k \neq 0$, $j(k)$ is an even function $j(-k) = j(k)$ with discontinuity on the real line for $\operatorname{Im} k = 0$ with jump

$$j(k) - j(-k), \quad \operatorname{Im} k = 0.$$

Using (116), we find that

$$j(k) = -\frac{1}{\mathrm{i}k}E(-\mathrm{i}k) - \mathbf{M}_+(k)Y_1^{-1}. \tag{141}$$

Consequently, $j(k) = \mathscr{O}\left(\frac{1}{|k|^2}\right)$ and $j(k) - j(-k) = \mathscr{O}\left(\frac{1}{|k|^3}\right)$, $k\in\mathbb{C}_+$. The Weyl matrix directly determines $j$; this is because the two leading terms in the asymptotic expansion of $\mathbf{M}$ determine $E$ and $Y_1$ [cf. (114)–(117)].

As $T_0Y_0 = Y_0$, we note that

$$\boldsymbol{\phi}_0(x,k) - T_0e(x,k) = \mathscr{O}(1) \tag{142}$$

[cf. (132)].

*Proposition VII.1. The function $\Psi(x,k)$ has asymptotic expansion*

$$\Psi(x,k) = T_0 + \frac{1}{\mathrm{i}k}\left\{\left(\mathbf{D}(x) + \frac{\omega^2}{2\mu_0}x\right)T_0 - T_1\right\} + o\left(\frac{1}{|k|}\right), \quad \operatorname{Im} k \le 0, \tag{143}$$

*where $\mathbf{D}(x)$ is given in Lemma VII.1 and $T_0, T_1$ are defined by (118). It admits the representation*

$$\begin{aligned}\Psi(x,k) = T_0 &- \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{k'e^{\mathrm{i}k'x}\widehat{\boldsymbol{\varphi}}(x,k')(j(k') - j(-k'))E(-\mathrm{i}k')}{k'-k}\mathrm{d}k'\\ &+ \sum_{j=1}^{N}\left(\frac{e^{\mathrm{i}k_jx}\boldsymbol{\varphi}(x,k_j)C_jE(-\mathrm{i}k_j)}{2\mathrm{i}(k-k_j)} + \frac{e^{-\mathrm{i}k_jx}\boldsymbol{\varphi}(x,k_j)C_jE(\mathrm{i}k_j)}{2\mathrm{i}(k+k_j)}\right).\end{aligned} \tag{144}$$

*Moreover, the boundary values* $\Psi_\pm(x,k) = \Psi(x, k \pm \mathrm{i}0)$, $\mathrm{Im}\, k = 0$, *determine* $\boldsymbol{\varphi}$ *in (70) by*

$$2\boldsymbol{\varphi}(x,k) = e^{-\mathrm{i}kx}\Psi_+(x,k)E(-\mathrm{i}k)^{-1} + e^{\mathrm{i}kx}\Psi_-(x,-k)E(\mathrm{i}k)^{-1}. \tag{145}$$

*Proof.* Substituting (118) and (119) into Corollary VII.1 yields (143) for $\mathrm{Im}\, k < 0$, and by taking a one-sided limit, we get (143) for $\mathrm{Im}\, k = 0$. For real-valued $k$, $\mathrm{Im}\, k = 0$, an explicit calculation gives

$$\Psi(x, k + \mathrm{i}\,0) - \Psi(x, k - \mathrm{i}0) = -\mathrm{i}ke^{\mathrm{i}kx}\boldsymbol{\varphi}(x,k)(j(k) - j(-k))E(-\mathrm{i}k). \tag{146}$$

Hence,

$$\begin{aligned}\Psi(x, k + \mathrm{i}\,0) + \mathrm{i}ke^{\mathrm{i}kx}\boldsymbol{\phi}_0(x,k)(j(k) - j(-k))E(-\mathrm{i}k) - \Psi(x, k - \mathrm{i}\,0)\\ = -\mathrm{i}ke^{\mathrm{i}kx}\widehat{\boldsymbol{\varphi}}(x,k)(j(k) - j(-k))E(-\mathrm{i}k) = \mathscr{O}\left(\frac{1}{|k|}\right)\end{aligned} \tag{147}$$

[cf. (134)]. This implies that the leading order in asymptotics (143) is the same for $\Psi(x, k - \mathrm{i}0)$ and $\Psi(x, k + \mathrm{i}0) + \mathrm{i}k\boldsymbol{\phi}_0(x,k)(j(k) - j(-k))E(-\mathrm{i}k)$ in the full complex $k$-plane and also (as $\Psi$ is bounded on $\mathbb{C}$) that $\Psi$ can be recovered from the Cauchy integral formula and its residues.

Now, we prove representation (144). We let $R > 0$ and denote by $\gamma_R^\pm$ the closed half-circle contour in $\mathbb{C}_\pm$ with positive orientation containing all the poles $\pm k_j$ inside. We denote by $\Gamma_R^\pm$ only the arc parts of these contours with negative orientation. We have

$$\begin{aligned}&\frac{1}{2\pi\mathrm{i}}\int_{-R}^{R}\frac{\Psi_+(x,k') + \mathrm{i}k'e^{\mathrm{i}k'x}\boldsymbol{\phi}_0(x,k')(j(k') - j(-k'))E(-\mathrm{i}k') - \Psi_-(x,k')}{k-k'}\mathrm{d}k'\\ &\quad = \mathrm{Er}_R(x,k) - \Psi(x,k) + \mathbf{D}_0(x)T_0 + \sum_{j=1}^{N}\frac{\mathrm{Res}_{k'=k_j}(\Psi(x,k') + \mathrm{i}k'e^{\mathrm{i}k'x}\boldsymbol{\phi}_0(x,k')(j(k') - j(-k'))E(-\mathrm{i}k'))}{k-k_j}\\ &\quad + \sum_{j=1}^{N}\frac{\mathrm{Res}_{k'=-k_j}\Psi(x,k')}{k'+k_j},\end{aligned}$$

where

$$\mathrm{Er}_R(x,k) = \frac{1}{2\pi\mathrm{i}}\int_{\Gamma_R^+}\frac{\Psi_+(x,k') + \mathrm{i}k'e^{\mathrm{i}k'x}\boldsymbol{\phi}_0(x,k')(j(k') - j(-k'))E(-\mathrm{i}k') - T_0}{k-k'}\mathrm{d}k' + \frac{1}{2\pi\mathrm{i}}\int_{\Gamma_R^-}\frac{\Psi_-(x,k') - T_0}{k-k'}\mathrm{d}k'.$$

Lemma VII.3 provides us with the expressions for the residues; we use that $j(k') - j(-k')$ is entire for $\mathrm{Im}\, k \neq 0$.

We substitute (147) in the integrand of the integral in the left-hand side and take the limit $R \to \infty$. As

$$\lim_{R\to\infty}\mathrm{Er}_R(x,k) = 0,$$

we obtain (144).

Finally, using (116) and that $\boldsymbol{\Phi}_-(x,-k) = \boldsymbol{\Phi}_+(x,k)$ for real-valued $k$ by definition, we obtain (145). □

We factorize $\boldsymbol{\varphi}$ and $\widehat{\boldsymbol{\varphi}}$ and introduce $\mathscr{A}$ and $\widehat{\mathscr{A}}$ according to

$$\boldsymbol{\varphi}(x,k) = T_0\mathscr{A}(x,k), \quad \widehat{\boldsymbol{\varphi}}(x,k) = T_0\widehat{\mathscr{A}}(x,k), \quad k \in \mathbb{R}, \tag{148}$$

and write

$$\mathscr{A}_j(x) = \mathscr{A}(x,k_j), \quad j = 1,\ldots,N.$$

We note that

$$\mathrm{Res}_{\pm k_j\in\mathbb{C}_\pm}\widehat{\mathscr{A}}(x,k)j(k) = \mathrm{Res}_{\pm k_j\in\mathbb{C}_\pm}\mathscr{A}(x,k)j(k). \tag{149}$$

We introduce

$$\mathring{e}(x,k) = T_0^{-1}\left\{T_0 e(x,k) - \boldsymbol{\phi}_0(x,k)\left(1 + \frac{1}{2}\mathrm{i}k(j(k) - j(-k))\right)\right\} \tag{150}$$

[cf. (132)] in the following.

*Lemma VII.4. The functions* $\mathscr{A}(x,k)$, $\widehat{\mathscr{A}}(x,k)$, *and* $\mathscr{A}_j(x)$, $j = 1,\ldots,N$, *satisfy*

$$4\widehat{\mathscr{A}}(x,k) = 4\overset{\circ}{e}(x,k) + \frac{1}{\pi \mathrm{i}}\int_{-\infty}^{\infty} k'\,\widehat{\mathscr{A}}(x,k')j(k')\big(\tilde{e}(x,k'-k) + \tilde{e}(x,k'+k)\big)\mathrm{d}k' - \sum_{j=1}^{N}\mathscr{A}_j(x)C_j(\tilde{e}(x,k_j-k) + \tilde{e}(x,k_j+k)), \quad \mathrm{Im}\,k = 0. \tag{151}$$

*This equation holds, in particular, for* $k = k_i$, $i = 1,\ldots,N$.

*Proof.* We take (144) as a point of departure. We first study the behavior of this equality, upon multiplication by $\mathrm{i}k\,T_0^{-1}$, as $k \to \infty$, in particular, the integral contribution (suppressing the factor $\frac{1}{2\pi}$) in the right-hand side,

$$\int_{-\infty}^{\infty}\frac{k' e^{\mathrm{i}k'x}\widehat{\boldsymbol{\varphi}}(x,k')(j(k') - j(-k'))E(-\mathrm{i}k')}{k'-k}\mathrm{d}k'.$$

To establish uniform boundedness, it is sufficient to only consider the "leading" order of $\widehat{\boldsymbol{\varphi}}(x,k')$, that is, the second term in (131),

$$k' e^{\mathrm{i}k'x}(e^{\mathrm{i}k'x} + e^{-\mathrm{i}k'x})\mathbf{m_1}(x)(j(k') - j(-k'))\begin{pmatrix} 0 & 0 \\ -2\varpi\mathrm{i}k' & 0\end{pmatrix} \tag{152}$$

or, in fact, the non-vanishing matrix element (suppressing the factor $-2\varpi\mathrm{i}$),

$$(k')^2 e^{2\mathrm{i}k'x}\mathbf{m_1}(x)(j(k') - j(-k')) + (k')^2\mathbf{m_1}(x)(j(k') - j(-k')),$$

in the numerator of the integrand. Concerning the second term, we note that

$$0 = \int_{-\infty}^{\infty}(k')^2\mathbf{m_1}(x)(j(k') - j(-k'))\mathrm{d}k' = \lim_{R\to\infty}\lim_{k\to\infty}(-k)\int_{-R}^{R}\frac{(k')^2\mathbf{m_1}(x)(j(k') - j(-k'))}{k'-k}\mathrm{d}k', \tag{153}$$

which holds true also for $x = 0$. Hence,

$$\lim_{R\to\infty}\lim_{k\to\infty}(-\mathrm{i})kT_0^{-1}\int_{-R}^{R}\frac{k'\widehat{\boldsymbol{\varphi}}(0,k')(j(k') - j(-k'))E(-\mathrm{i}k')}{k'-k}\mathrm{d}k' = 0. \tag{154}$$

We now let $x > 0$ and perform integration by parts, exploiting the exponent $e^{2\mathrm{i}k'x}$, while analyzing the first term,

$$\int_{-R}^{R}\frac{(k')^2\mathbf{m_1}(x)e^{2\mathrm{i}k'x}(j(k') - j(-k'))}{k'-k}\mathrm{d}k' = -\int_{-R}^{R}\frac{\mathbf{m_1}(x)}{2\mathrm{i}x}e^{2\mathrm{i}k'x}\frac{\frac{\mathrm{d}}{\mathrm{d}k'}\{(k')^2(j(k') - j(-k'))\}}{k'-k}\mathrm{d}k' + \mathrm{Er}_{R;1}(x,k) + \int_{-R}^{R}\frac{\mathbf{m_1}(x)}{2\mathrm{i}x}e^{2\mathrm{i}k'x}\frac{(k')^2(j(k') - j(-k'))}{(k'-k)^2}\mathrm{d}k' + \mathrm{Er}_{R;2}(x,k).$$

We have

$$\lim_{R\to\infty}\lim_{k\to\infty}(-\mathrm{i})k\mathrm{Er}_{R;1,2}(x,k) = 0,$$

whence

$$\lim_{R\to\infty}\lim_{k\to\infty}(-\mathrm{i})k\int_{-R}^{R}\frac{(k')^2\mathbf{m_1}(x)e^{2\mathrm{i}k'x}(j(k') - j(-k'))}{k'-k}\mathrm{d}k' = \lim_{R\to\infty}(-\mathrm{i})\int_{-R}^{R}\frac{\mathbf{m_1}(x)}{2\mathrm{i}x}e^{2\mathrm{i}k'x}\frac{\mathrm{d}}{\mathrm{d}k'}\{(k')^2(j(k') - j(-k'))\}\mathrm{d}k' \quad \text{is bounded.} \tag{155}$$

This implies that

$$\lim_{R\to\infty}\lim_{k\to\infty}(-\mathrm{i})kT_0^{-1}\int_{-R}^{R}\frac{k' e^{\mathrm{i}k'x}\widehat{\boldsymbol{\varphi}}(x,k')(j(k') - j(-k'))E(-\mathrm{i}k')}{k'-k}\mathrm{d}k' = \lim_{R\to\infty}\int_{-R}^{R}\mathrm{i}k' e^{\mathrm{i}k'x}T_0^{-1}\widehat{\boldsymbol{\varphi}}(x,k')(j(k') - j(-k'))E(-\mathrm{i}k')\mathrm{d}k' = 2\int_{-\infty}^{\infty}\mathrm{i}k'\,\widehat{\mathscr{A}}(x,k')j(k')e(x,k')\mathrm{d}k' \quad \text{is bounded.} \tag{156}$$

Here, we used that the integrand is even in $k'$.

We use (145) and that $\boldsymbol{\varphi}$ is even in $k$ to get

$$4\boldsymbol{\varphi}(x,k) = e^{-\mathrm{i}kx}(\Psi_+(x,k)+\Psi_-(x,k))E(-\mathrm{i}k)^{-1} + e^{\mathrm{i}kx}(\Psi_-(x,-k)+\Psi_+(x,-k))E(\mathrm{i}k)^{-1}.$$

Adding and subtracting terms, we rewrite this equality as

$$4\boldsymbol{\varphi}(x,k) = 4T_0 e(x,k) - 2\mathrm{i}k\boldsymbol{\phi}_0(x,k)(j(k)-j(-k)) + \mathbf{I}_1(x,k)+\mathbf{I}_2(x,k)+\mathbf{I}_3(x,k)+\mathbf{I}_4(x,k) \tag{157}$$

or

$$4\widehat{\boldsymbol{\varphi}}(x,k) = 4T_0\mathring{e}(x,k) + \mathbf{I}_1(x,k)+\mathbf{I}_2(x,k)+\mathbf{I}_3(x,k)+\mathbf{I}_4(x,k), \tag{158}$$

where

$$\begin{aligned}
\mathbf{I}_1 &= e^{-\mathrm{i}kx}(\Psi_+(x,k)+\mathrm{i}ke^{\mathrm{i}kx}\boldsymbol{\phi}_0(x,k)(j(k)-j(-k))E(-\mathrm{i}k)-T_0)E^{-1}(-\mathrm{i}k),\\
\mathbf{I}_2 &= e^{-\mathrm{i}kx}(\Psi_-(x,k)-T_0)E^{-1}(-\mathrm{i}k),\\
\mathbf{I}_3 &= e^{\mathrm{i}kx}(\Psi_+(x,-k)+\mathrm{i}ke^{-\mathrm{i}kx}\boldsymbol{\phi}_0(x,k)(j(k)-j(-k))E(\mathrm{i}k)-T_0)E^{-1}(\mathrm{i}k),\\
\mathbf{I}_4 &= e^{\mathrm{i}kx}(\Psi_-(x,-k)-T_0)E^{-1}(\mathrm{i}k).
\end{aligned}$$

We note that $\mathbf{I}_1$ and $\mathbf{I}_4$ have poles, $k_j$, in the complex $k$ half plane $\mathbb{C}_+$ $(\operatorname{Im} k > 0)$ and that $\mathbf{I}_2$ and $\mathbf{I}_3$ have poles, $-k_j$, in the complex $k$ half plane $\mathbb{C}_-$ $(\operatorname{Im} k < 0)$. We introduce a positive half-circle (with radius $R$) contour $\gamma_R^+$ in $\mathbb{C}_+$ containing poles $k_j$ and $k$ inside and a negative half-circle (with radius $R$) contour $\gamma_R^-$ in $\mathbb{C}_-$ containing poles $-k_j$ and $k$ inside. Then,

$$\begin{aligned}
&\frac{1}{2\pi\mathrm{i}}\lim_{R\to\infty}\int_{\gamma_R^+}\frac{e^{-\mathrm{i}kx}(\Psi_+(x,k')+\mathrm{i}k'e^{\mathrm{i}k'x}\boldsymbol{\phi}_0(x,k')(j(k')-j(-k'))E(-\mathrm{i}k')-T_0)E(-\mathrm{i}k)^{-1}}{k'-k}\mathrm{d}k'\\
&\quad= \mathbf{I}_1(x,k) + \sum_{j=1}^{N}\frac{e^{-\mathrm{i}kx}\operatorname{Res}_{k'=k_j}(\Psi_+(x,k')+\mathrm{i}k'e^{\mathrm{i}k'x}\boldsymbol{\phi}_0(x,k')(j(k')-j(-k'))E(-\mathrm{i}k'))E(-\mathrm{i}k)^{-1}}{k_j-k},
\end{aligned} \tag{159}$$

$$\frac{1}{2\pi\mathrm{i}}\lim_{R\to\infty}\int_{\gamma_R^-}\frac{e^{-\mathrm{i}kx}(\Psi_-(x,k')-T_0)E(-\mathrm{i}k)^{-1}}{k'-k}\mathrm{d}k' = \mathbf{I}_2(x,k) + \sum_{j=1}^{N}\frac{e^{-\mathrm{i}kx}\operatorname{Res}_{k'=-k_j}\Psi_-(x,k')E(-\mathrm{i}k)^{-1}}{-k_j-k}, \tag{160}$$

$$\begin{aligned}
&\frac{1}{2\pi\mathrm{i}}\lim_{R\to\infty}\int_{\gamma_R^-}\frac{e^{\mathrm{i}kx}(\Psi_+(x,-k')+\mathrm{i}k'e^{-\mathrm{i}k'x}\boldsymbol{\phi}_0(x,k')(j(k')-j(-k'))E(\mathrm{i}k')-T_0)E(\mathrm{i}k)^{-1}}{k'-k}\mathrm{d}k'\\
&\quad= \mathbf{I}_3(x,k) + \sum_{j=1}^{N}\frac{e^{\mathrm{i}kx}\operatorname{Res}_{k'=-k_j}(\Psi_+(x,-k')+\mathrm{i}k'e^{-\mathrm{i}k'x}\boldsymbol{\phi}_0(x,k')(j(k')-j(-k'))E(\mathrm{i}k'))E(\mathrm{i}k)^{-1}}{-k_j-k},
\end{aligned} \tag{161}$$

$$\frac{1}{2\pi\mathrm{i}}\lim_{R\to\infty}\int_{\gamma_R^+}\frac{e^{\mathrm{i}kx}(\Psi_-(x,-k')-T_0)E(\mathrm{i}k)^{-1}}{k'-k}\mathrm{d}k' = \mathbf{I}_4(x,k) + \sum_{j=1}^{N}\frac{e^{\mathrm{i}kx}\operatorname{Res}_{k'=k_j}\Psi_-(x,-k')E(\mathrm{i}k)^{-1}}{k_j-k}. \tag{162}$$

We observe that the contributions from the upper and lower semicircles in the limit vanish, that is,

$$\begin{aligned}
&\frac{1}{2\pi\mathrm{i}}\lim_{R\to\infty}\int_{\gamma_R^+}\frac{e^{-\mathrm{i}kx}(\Psi_+(x,k')+\mathrm{i}k'e^{\mathrm{i}k'x}\boldsymbol{\phi}_0(x,k')(j(k')-j(-k'))E(-\mathrm{i}k')-T_0)E(-\mathrm{i}k)^{-1}}{k'-k}\mathrm{d}k'\\
&\quad= \frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{\infty}\frac{e^{-\mathrm{i}kx}(\Psi_+(x,k')+\mathrm{i}k'e^{\mathrm{i}k'x}\boldsymbol{\phi}_0(x,k')(j(k')-j(-k'))E(-\mathrm{i}k')-T_0)E(-\mathrm{i}k)^{-1}}{k'-k}\mathrm{d}k',
\end{aligned} \tag{163}$$

$$\frac{1}{2\pi\mathrm{i}}\lim_{R\to\infty}\int_{\gamma_R^-}\frac{e^{-\mathrm{i}kx}(\Psi_-(x,k')-T_0)E(-\mathrm{i}k)^{-1}}{k'-k}\mathrm{d}k' = -\frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{\infty}\frac{e^{-\mathrm{i}kx}(\Psi_-(x,k')-T_0)E(-\mathrm{i}k)^{-1}}{k'-k}\mathrm{d}k', \tag{164}$$

$$\frac{1}{2\pi \mathrm{i}} \lim_{R\to\infty} \int_{\gamma_R^-} \frac{e^{\mathrm{i}kx}(\Psi_+(x,-k') + \mathrm{i}k' e^{-\mathrm{i}k'x}\phi_0(x,k')(j(k') - j(-k'))E(\mathrm{i}k') - T_0)E(\mathrm{i}k)^{-1}}{k'-k}\mathrm{d}k'$$
$$= -\frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{\infty} \frac{e^{\mathrm{i}kx}(\Psi_+(x,-k') + \mathrm{i}k' e^{-\mathrm{i}k'x}\phi_0(x,k')(j(k') - j(-k'))E(\mathrm{i}k') - T_0)E(\mathrm{i}k)^{-1}}{k'-k}\mathrm{d}k', \tag{165}$$

$$\frac{1}{2\pi \mathrm{i}} \lim_{R\to\infty} \int_{\gamma_R^+} \frac{e^{\mathrm{i}kx}(\Psi_-(x,-k') - T_0)E(\mathrm{i}k)^{-1}}{k'-k}\mathrm{d}k' = \frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{\infty} \frac{e^{\mathrm{i}kx}(\Psi_-(x,-k') - T_0)E(\mathrm{i}k)^{-1}}{k'-k}\mathrm{d}k'. \tag{166}$$

Hence, using that in the residues, the terms containing $\phi_0(x,k')(j(k') - j(-k'))$ do not contribute,

$$\begin{aligned}
&\mathbf{I}_1(x,k) + \mathbf{I}_2(x,k) + \mathbf{I}_3(x,k) + \mathbf{I}_4(x,k)\\
&= \frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{\infty} \frac{e^{-\mathrm{i}kx}(\Psi_+(x,k') + \mathrm{i}k' e^{\mathrm{i}k'x}\phi_0(x,k')(j(k') - j(-k'))E(-\mathrm{i}k') - \Psi_-(x,k'))E(-\mathrm{i}k)^{-1}}{k'-k}\mathrm{d}k'\\
&- \frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{\infty} \frac{e^{\mathrm{i}kx}(\Psi_+(x,-k') + \mathrm{i}k' e^{-\mathrm{i}k'x}\phi_0(x,k')(j(k') - j(-k'))E(\mathrm{i}k') - \Psi_-(x,-k'))E(\mathrm{i}k)^{-1}}{k'-k}\mathrm{d}k'\\
&- \sum_{j=1}^N \frac{e^{-\mathrm{i}kx}\mathrm{Res}_{k'=k_j}\Psi_+(x,k')E(-\mathrm{i}k)^{-1}}{k_j-k} + \sum_{j=1}^N \frac{e^{-\mathrm{i}kx}\mathrm{Res}_{k'=-k_j}\Psi_-(x,k')E(-\mathrm{i}k)^{-1}}{k_j+k}\\
&+ \sum_{j=1}^N \frac{e^{\mathrm{i}kx}\mathrm{Res}_{k'=-k_j}\Psi_+(x,-k')E(\mathrm{i}k)^{-1}}{k_j+k} - \sum_{j=1}^N \frac{e^{\mathrm{i}kx}\mathrm{Res}_{k'=k_j}\Psi_-(x,-k')E(-\mathrm{i}k)^{-1}}{k_j-k}.
\end{aligned} \tag{167}$$

Using Lemma VII.3, the summations over the poles in (167) add up to

$$\begin{aligned}
&- \sum_{j=1}^N \frac{e^{-\mathrm{i}kx}\mathrm{Res}_{k'=k_j}\Psi_+(x,k')E^{-1}(-\mathrm{i}k)}{k_j-k} + \sum_{j=1}^N \frac{e^{-\mathrm{i}kx}\mathrm{Res}_{k'=-k_j}\Psi_+(x,k')E^{-1}(-\mathrm{i}k)}{k_j+k}\\
&+ \sum_{j=1}^N \frac{e^{\mathrm{i}kx}\mathrm{Res}_{k'=-k_j}\Psi_+(x,-k')E^{-1}(\mathrm{i}k)}{k_j+k} - \sum_{j=1}^N \frac{e^{\mathrm{i}kx}\mathrm{Res}_{k'=k_j}\Psi_-(x,-k')E^{-1}(-\mathrm{i}k)}{k_j-k}.\\
&\quad = -\sum_{j=1}^N \boldsymbol{\varphi}(x,k_j)C_j(\tilde{e}(x,k_j-k) + \tilde{e}(x,k_j+k)).
\end{aligned} \tag{168}$$

Using (146), the integrals in (167) add up to

$$\begin{aligned}
&\frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{\infty} \frac{e^{-\mathrm{i}kx}(\Psi_+(x,k') + \mathrm{i}k' e^{\mathrm{i}k'x}\phi_0(x,k')(j(k') - j(-k'))E(-\mathrm{i}k') - \Psi_-(x,k'))E(-\mathrm{i}k)^{-1}}{k'-k}\mathrm{d}k'\\
&- \frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{\infty} \frac{e^{\mathrm{i}kx}(\Psi_+(x,-k') + \mathrm{i}k' e^{-\mathrm{i}k'x}\phi_0(x,k')(j(k') - j(-k'))E(\mathrm{i}k') - \Psi_-(x,-k'))E(\mathrm{i}k)^{-1}}{k'-k}\mathrm{d}k'\\
&= \frac{1}{\pi}\int_{-\infty}^{\infty} (-\mathrm{i}k')\widehat{\boldsymbol{\varphi}}(x,k')j(k')(\tilde{e}(x,k'-k) + \tilde{e}(x,k'+k))\mathrm{d}k'.
\end{aligned} \tag{169}$$

We established boundedness of this integral in (156). Substituting (168) and (169) into (167) and the result into (158) implies the statements upon considering $k \in \mathbb{R}$ (and $k = k_j$). □

With Proposition VII.1 and the proof of the previous lemma concerning the limit $k \to \infty$ ($\mathrm{Im}\, k = 0$), we obtain the following.

*Lemma VII.5. The following holds true:*

$$T_0^{-1}\left\{\left(\mathbf{D}(x) + \frac{\omega^2}{2\mu_0}x\right)T_0 - T_1\right\} = -\frac{1}{\pi}\int_{-\infty}^{\infty} \mathrm{i}k' \widehat{\mathscr{A}}(x,k')j(k')e(x,k')\mathrm{d}k' + \sum_{j=1}^N \mathscr{A}_j(x)C_j e(x,k_j), \tag{170}$$

*where* $\mathbf{D}(x)$ *is given in Lemma VII.1.*

In the above, we note that $T_1$ depends on $\omega$ through $\theta_1$. This lemma provides an identity for $\mathbf{D}$. *It is essential for this lemma that we analyzed the asymptotic expansions of* $\mathbf{F}$, $\mathbf{M}$, *and* $\mathbf{\Phi}$. The right-hand side of (170) motivates the introduction of

$$K(x,y) = -\frac{1}{\pi}\int_{-\infty}^{\infty} \mathrm{i}k' \widehat{\mathscr{A}}(x,k')j(k')e(y,k')\mathrm{d}k' + \sum_{j=1}^{N}\mathscr{A}_j(x)C_j e(y,k_j)$$
$$= -\frac{1}{2\pi \mathrm{i}}\int_{-\infty}^{\infty} \widehat{\mathscr{A}}(x,k')(j(k')-j(-k'))e^{\mathrm{i}k'y}E(-\mathrm{i}k')k'\mathrm{d}k' + \sum_{j=1}^{N}\mathscr{A}_j(x)C_j e(y,k_j), \quad y\in[-x,x] \tag{171}$$

[cf. (139)], the first term of which can be identified with a Fourier transform and plays a key role in the proof of the next proposition. The right-hand of (170) is $K(x,x)$.

*Remark VII.1. From the analysis leading to (156), it follows that* $K(x,x)$ *is bounded. In fact, the continuous differentiability of* $K(x,x)$ *is directly related to the continuous differentiability of* $V$ *through* $\mathbf{D}(x)$.

We note that $T_0$ can be obtained from the asymptotic expansion of $\mathbf{M}$ [cf. (109) and (118)]. Suppose that $K(x,x)$ were known; then, the potential, $V$, can be recovered upon differentiating (170),

$$T_0^{-1}\mathbf{D}'(x)T_0^{-1} + \frac{\omega^2}{2\mu_0} = K'(x,x),$$

where $\mathbf{D}$ is given in Lemma VII.1 with

$$\mathbf{D}'(x) = \frac{1}{2}V(x)\begin{pmatrix} -G_{11}^H\left(\frac{c_0}{2}G_{12}^H H + G_{22}^H\right) & G_{11}^H\left(\frac{c_0}{2}G_{11}^H H + G_{21}^H\right) \\ -G_{12}^H\left(\frac{c_0}{2}G_{12}^H H + G_{22}^H\right) & G_{12}^H\left(\frac{c_0}{2}G_{11}^H H + G_{21}^H\right)\end{pmatrix}.$$

The kernel, $K(x,y)$, is determined by the boundary spectral data, which is the content of the following.

*Proposition VII.2 (Gel'fand-Levitan). The kernel* $K(x,y)$ *is the unique solution of the Gel'fand-Levitan type equation,*

$$4\ K(x,y) + 4\widehat{g}(x,y) - \int_{-x}^{x} K(x,y')E(2\delta(x+y'))g(-y',y)\mathrm{d}y' = 0, \quad y\in[-x,x], \tag{172}$$

*where* $E$ *is given in (117),*

$$g(x,y) = \frac{1}{\pi \mathrm{i}}\int_{-\infty}^{\infty} e(x,k)j(k)e(y,k)k\mathrm{d}k - \sum_{j=1}^{N} e(x,k_j)C_j e(y,k_j), \tag{173}$$

*and*

$$\widehat{g}(x,y) = \frac{1}{\pi \mathrm{i}}\int_{-\infty}^{\infty} \mathring{e}(x,k)j(k)e(y,k)k\mathrm{d}k - \sum_{j=1}^{N} \overset{*}{e}(x,k_j)C_j e(y,k_j), \tag{174}$$

*in which*

$$\overset{*}{e}(x,k) = e(x,k) - \frac{1}{2}\mathrm{i}kT_0^{-1}\phi_0(x,k)(j(k)-j(-k)). \tag{175}$$

*Proof.* We distinguish two parts to complete the proof. **Part I: construction of (172).** We consider (171) and write

$$K(x,y) = T_+(x,y) + T_-(x,-y),$$

where

$$T_\pm(x,y) = -\frac{1}{2\pi \mathrm{i}}\int_{-\infty}^{\infty} \widehat{\mathscr{A}}(x,k)j(k)e^{\mathrm{i}ky}E(\mp \mathrm{i}k)k\mathrm{d}k + \frac{1}{2}\sum_{j=1}^{N}\mathscr{A}_j(x)C_j e^{\mathrm{i}k_j y}E(\mp \mathrm{i}k_j).$$

We note that [cf. (149)]

$$T_\pm(x,y) = -\frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{\infty}\widehat{A}(x,k)j(k)e^{\mathrm{i}ky}E(\mp\mathrm{i}k)k\mathrm{d}k + \sum_j \mathrm{Res}_{k_j\in\mathbb{C}_+}\mathscr{A}(x,k)j(k)e^{\mathrm{i}ky}E(\mp\mathrm{i}k)k$$
$$= \lim_{R\to\infty}\frac{1}{2\pi\mathrm{i}}\int_{\Gamma_R^+}\widehat{\mathscr{A}}(x,k)e^{\mathrm{i}kx}j(k)e^{\mathrm{i}k(y-x)}E(\mp\mathrm{i}k)k\mathrm{d}k \tag{176}$$

and

$$T_\pm(x,y) = -\frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{\infty}\widehat{\mathscr{A}}(x,k)j(k)e^{\mathrm{i}ky}E(\mp\mathrm{i}k)k\mathrm{d}k - \sum_j \mathrm{Res}_{-k_j\in\mathbb{C}_-}\widehat{\mathscr{A}}(x,k)j(k)e^{\mathrm{i}ky}E(\mp\mathrm{i}k)k$$
$$= -\lim_{R\to\infty}\frac{1}{2\pi\mathrm{i}}\int_{\Gamma_R^-}\widehat{\mathscr{A}}(x,k)e^{-\mathrm{i}kx}j(k)e^{\mathrm{i}k(y+x)}E(\mp\mathrm{i}k)k\mathrm{d}k \tag{177}$$

with

$$-\sum_j \mathrm{Res}_{-k_j\in\mathbb{C}_-}\widehat{\mathscr{A}}(x,k)j(k)e^{\mathrm{i}ky}E(\mp\mathrm{i}k)k = \sum_j \mathrm{Res}_{k_j\in\mathbb{C}_+}\widehat{\mathscr{A}}(x,k)j(k)e^{\mathrm{i}ky}E(\mp\mathrm{i}k)k \tag{178}$$

as

$$\lim_{k\to k_j}(k-k_j)j(k) = -\lim_{k\to -k_j}(k+k_j)j(k).$$

The absence of singularities means that $\mathscr{A}(x,k)e^{\pm\mathrm{i}kx}$ has a bounded holomorphic extension to the half-plane $\pm\mathrm{Im}\,k \geq 0$. Through the exponential decay of $e^{\mathrm{i}k(y-x)}$, we find that

$$\text{if } |y| > x, \text{ then } T_\pm(x,y) = 0.$$

From the Fourier inversion formula, we obtain

$$\widehat{\mathscr{A}}(x,k)j(k)k - \pi\mathrm{i}\sum_{j=1}^{N}\mathscr{A}_j(x)C_j\delta(k-k_j) = -\left(\mathrm{i}\int_{-x}^{x}T_\pm(x,y)e^{-\mathrm{i}ky}\mathrm{d}y\right)E(\pm\mathrm{i}k).$$

We substitute this expression in (151). Using that

$$\frac{1}{\pi\mathrm{i}}\int_{-\infty}^{\infty}k'\left[-\pi\mathrm{i}\sum_{j=1}^{N}\mathscr{A}_j(x)C_j\delta(k-k_j)\right]j(k')(\tilde{e}(x,k'-k)+\tilde{e}(x,k'+k))\mathrm{d}k' = \sum_{j=1}^{N}\mathscr{A}_j(x)C_j(\tilde{e}(x,k_j-k)+\tilde{e}(x,k_j+k)),$$

which shows that the summation over poles in (151) is cancelled, we obtain

$$4\widehat{\mathscr{A}}(x,k) = 4\mathring{e}(x,k) - \int_{-x}^{x}T_\pm(x,y)B_\pm(k,y)\mathrm{d}y, \tag{179}$$

in which

$$B_\pm(k,y) = \frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{\infty}e^{-\mathrm{i}k'y}E(\pm\mathrm{i}k')\left(\frac{e^{\mathrm{i}(k'-k)x}E(-\mathrm{i}(k'-k))}{k'-k} - \frac{e^{-\mathrm{i}(k'-k)x}E(\mathrm{i}(k'-k))}{k'-k}\right.$$
$$\left. + \frac{e^{\mathrm{i}(k'+k)x}E(-\mathrm{i}(k'+k))}{k'+k} - \frac{e^{-\mathrm{i}(k'+k)x}E(\mathrm{i}(k'+k))}{k'+k}\right)\mathrm{d}k'.$$

By straightforward calculations, we find that

$$B_+(k,y) = (\mathrm{sgn}(x-y)e(y,k) + \mathrm{sgn}(x+y)e(-y,k))$$
$$+ 8\varpi\delta(x+y)e(-y,k)\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix} + 4\varpi\,\mathrm{sgn}(x+y)\frac{\mathrm{d}}{\mathrm{d}y}e(-y,k)\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix} \tag{180}$$

and

$$B_-(k,y) = B_+(k,-y) = (\mathrm{sgn}(x+y)e(-y,k) + \mathrm{sgn}(x-y)e(y,k))$$
$$+ 8\varpi\delta(x-y)e(y,k)\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix} + 4\varpi\,\mathrm{sgn}(x-y)\frac{\mathrm{d}}{\mathrm{d}y}e(y,k)\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}. \tag{181}$$

Taking half the sum of the ± representations in (179), we get

$$4\hat{\mathscr{A}}(x,k) - 4\overset{\circ}{e}(x,k) = -\frac{1}{2}\int_{-x}^{x} K(x,y)B_+(k,y)\mathrm{d}y.$$

Substituting (180) and (181) into this equation and using that $e(x,k) + e(-x,k) = 2\cos kx\, I_2$, we get

$$4\hat{\mathscr{A}}(x,k) - 4\overset{\circ}{e}(x,k) = -\int_{-x}^{x} K(x,y')E(2\delta(x+y'))e(-y',k)\mathrm{d}y'. \tag{182}$$

We multiply this equation by $2j(k)e(y,k)k - 2\pi\mathrm{i}\sum_{j=1}^{N}\delta(k-k_j)C_je(y,k_j)$ and obtain

$$\begin{aligned}
&4\hat{\mathscr{A}}(x,k)j(k)2e(y,k)k - 8\pi\mathrm{i}\mathscr{A}(x,k)\sum_{j=1}^{N}\delta(k-k_j)C_je(y,k_j)\\
&- 4\overset{\circ}{e}(x,k)j(k)2e(y,k)k + 8\pi\mathrm{i}\overset{*}{e}(x,k)\sum_{j=}^{N}\delta(k-k_j)C_je(y,k_j)\\
&= -\int_{-x}^{x} K(x,y')E(2\delta(x+y'))e(-y',k)\mathrm{d}y'\left(j(k)2e(y,k)k - 2\pi\mathrm{i}\sum_{j=1}^{N}\delta(k-k_j)C_je(y,k_j)\right).
\end{aligned}$$

Dividing this equation by $2\pi\mathrm{i}$ and integrating over $k$ lead to (172). More precisely, we first integrate over $[-R,R]$ and establish that the integrals are uniformly bounded after which we interchange orders of integration and take the limit $R\to\infty$.

As

$$\int_{-x}^{x}\begin{pmatrix}0 & 0\\ 2\varpi k\,\sin(ky') & 0\end{pmatrix}\left(I_2 + \begin{pmatrix}0 & 0\\ 4\varpi\delta(x+y') & 0\end{pmatrix}\right)\cos(k'y')\mathrm{d}y' = 0$$

the representation of $g$, in fact, can be simplified,

$$g(x,y) = \frac{1}{\pi\mathrm{i}}\int_{-\infty}^{\infty}\cos(kx)j(k)e(y,k)k\mathrm{d}k - \sum_{j=1}^{N}\cos(k_jx)C_je(y,k_j). \tag{183}$$

**Part II: (172) has a unique solution.** We note that $g(x,y)=0$ for $|y|>x$ and that Eq. (172) is of Volterra type. We consider $x$ as parameter and $K(x,y)$ as unknown function. For unique solvability, we need to prove that, for some constant $C>0$ (dependent on $x$),

$$\sup_{|y|\le x}\int_{-x}^{x}|E(2\delta(x+y'))g(-y',y)|\mathrm{d}y' \le C. \tag{184}$$

Using a special form of matrices $g$ and $E$, it follows that (184) is satisfied. Then, using the Volterra property, it follows that the solution to the homogeneous problem is trivial and the solution to (172) can be constructed by iteration. This completes the Proof of Proposition V.1. □

## B. Recovery of $G^H$

Here, we prove that $G^H$ is determined by the two leading orders in asymptotic expansion of the Jost solution $\mathbf{F}$ at $x=0$ as $\xi\to\infty$, $\xi\in\mathscr{K}_+$. The asymptotic expansion of $\mathbf{F}(0,\xi)$ is given by (101)–(103) upon substituting $x=0$,

$$\begin{aligned}
\mathbf{F}(0,\xi) = \xi\Bigg(&-\frac{\mu_0}{\omega^2}\begin{pmatrix}G_{11}^H & G_{11}^H\\ G_{12}^H & G_{12}^H\end{pmatrix}\\
&+\frac{1}{\xi}\left\{\begin{pmatrix}G_{11}^H\dfrac{(\lambda_0+\mu_0)H}{2(\lambda_0+2\mu_0)} + G_{21}^H & 0\\ G_{12}^H\dfrac{(\lambda_0+\mu_0)H}{2(\lambda_0+2\mu_0)} + G_{22}^H & 0\end{pmatrix} - \frac{1}{2}\frac{\mu_0}{\omega^2}\int_0^H V(y)\mathrm{d}y\begin{pmatrix}G_{11}^H & G_{11}^H\\ G_{12}^H & G_{12}^H\end{pmatrix}\right\}\\
&+o\left(\frac{1}{|\xi|}\right)\Bigg).
\end{aligned}$$

The expression for $V(y)$ can be directly deduced from the analysis in Sec. III. From the leading order term in this asymptotic expansion, we recover $G_{11}^H$ and $G_{12}^H$.

In the next order term, we write as $S(\omega) = X + \omega^{-2}Y$, where $X$ and $Y$ are independent of $\omega$ and given by

$$X = \begin{pmatrix} G_{11}^H \frac{(\lambda_0+\mu_0)H}{2(\lambda_0+2\mu_0)} + G_{21}^H & 0 \\ G_{12}^H \frac{(\lambda_0+\mu_0)H}{2(\lambda_0+2\mu_0)} + G_{22}^H & 0 \end{pmatrix} - \frac{\mu_0}{2}\int_0^H \Bigg( \big(G^{-1}(y)B_2(x)G(x)\big)^{\mathrm{T}} - \left( G_0^{-1}(y) \begin{pmatrix} -\frac{1}{\mu_0} & 0 \\ 0 & -\frac{1}{\lambda_0+2\mu_0} \end{pmatrix} G_0(x) \right)^{\mathrm{T}} \Bigg) \mathrm{d}y \begin{pmatrix} G_{11}^H & G_{11}^H \\ G_{12}^H & G_{12}^H \end{pmatrix} \tag{185}$$

and

$$Y = -\frac{1}{2}\mu_0 \int_0^H \big(G^{-1}B_1 G\big)^{\mathrm{T}} \mathrm{d}y \begin{pmatrix} G_{11}^H & G_{11}^H \\ G_{12}^H & G_{12}^H \end{pmatrix}. \tag{186}$$

Using $S(\omega_1), S(\omega_2)$ for any two frequencies $\omega_1 \neq \omega_2 \in \mathbb{R}_+$, we get

$$Y = \frac{1}{\frac{1}{\omega_1^2} - \frac{1}{\omega_2^2}} (S(\omega_1) - S(\omega_2))$$

and, then, simply, $X = S(\omega_1) - \omega_1^{-2}Y$. We multiply $X$ from the right with $\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$ and obtain

$$X\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} G_{11}^H \frac{(\lambda_0+\mu_0)H}{2(\lambda_0+2\mu_0)} & G_{11}^H \frac{(\lambda_0+\mu_0)H}{2(\lambda_0+2\mu_0)} \\ G_{12}^H \frac{(\lambda_0+\mu_0)H}{2(\lambda_0+2\mu_0)} & G_{12}^H \frac{(\lambda_0+\mu_0)H}{2(\lambda_0+2\mu_0)} \end{pmatrix} + \begin{pmatrix} G_{21}^H & G_{21}^H \\ G_{22}^H & G_{22}^H \end{pmatrix}. \tag{187}$$

As we already recovered $G_{11}^H$ and $G_{12}^H$ and $G_{11}^H G_{22}^H - G_{12}^H G_{21}^H = 1$, we obtain $G_{21}^H$ and $G_{22}^H$.

### C. Recovery of $\lambda$ and $\mu$

With the recovery of $G^H$, we recover $Q_0$ and, hence, $Q$. Finally, we note that $Q = Q(\omega) = Q_1 + \omega^2 Q_2$ with $Q_j$ related to $B_j^{\mathrm{T}}$ in (21) and (22) by similarity transformations. Then, if $Q(\omega_1), Q(\omega_2)$ are known for some frequencies $\omega_1 \neq \omega_2 \in \mathbb{R}_+$, we obtain

$$Q_2 = \frac{1}{\omega_1^2 - \omega_2^2}(Q(\omega_1) - Q(\omega_2))$$

and, then,

$$\operatorname{Tr} Q_2 = -\frac{1}{\mu} - \frac{1}{\lambda + 2\,\mu} \quad \text{and} \quad \det Q_2 = \frac{1}{\mu}\,\frac{1}{\lambda + 2\,\mu},$$

wherefrom $\lambda$ and $\mu$ are recovered.

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## AUTHOR DECLARATIONS

### Conflict of Interest

The authors have no conflicts to disclose.

## DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

## APPENDIX A: RIEMANN SURFACE

For the introduction of the proper Riemann surface, we refer to the work of Chapman.[21] We denote by $\sqrt{z}$ the principal branch of the square root that is positive for $z > 0$ and with the cut along the negative real axis. For the analytic continuation in $|\xi| \in \mathbb{R}_+$, we replace $|\xi|$ by $\xi \in \mathbb{C}$. We define $q_S(\xi)$ by choosing the branch with

$$q_S(\xi) \in \mathrm{i}\mathbb{R}_+ \ \text{for real-valued } \xi > \frac{\omega}{\sqrt{\mu_0}} \quad \text{and} \quad q_S(\xi) \in \mathrm{i}\mathbb{R}_- \ \text{for real-valued } \xi < -\frac{\omega}{\sqrt{\mu_0}}.$$

Then,

$$\operatorname{Im} q_S(\xi) > 0 \quad \text{for } \operatorname{Re}\xi > \frac{\omega}{\sqrt{\mu_0}} \quad \text{and} \quad \operatorname{Im} q_S(\xi) < 0 \quad \text{for } \operatorname{Re}\xi < -\frac{\omega}{\sqrt{\mu_0}}.$$

We note that $\operatorname{Im} q_S(\xi) = 0$ for $\xi \in \left[-\frac{\omega}{\sqrt{\mu_0}}, \frac{\omega}{\sqrt{\mu_0}}\right] \cup \mathrm{i}\mathbb{R}$.

We let[22]

$$\mathscr{K}_S := \mathbb{C}\setminus\left(\left[-\frac{\omega}{\sqrt{\mu_0}}, \frac{\omega}{\sqrt{\mu_0}}\right] \cup \mathrm{i}\mathbb{R}\right).$$

We observe that $q_S: \ \xi \to \sqrt{\frac{\omega^2}{\mu_0} - \xi^2}$ is a conformal mapping $\mathscr{K}_S \to \mathscr{K}_S$ and satisfies

$$q_S(\xi) = \mathrm{i}\xi - \frac{\mathrm{i}\omega^2}{2\mu_0\xi} + \mathscr{O}\left(\frac{1}{|\xi|^2}\right) \qquad \text{as} \quad |\xi| \to \infty. \tag{A1}$$

Moreover, $q_S$ maps the cut $[-\frac{\omega}{\sqrt{\mu_0}}, \frac{\omega}{\sqrt{\mu_0}}]$ on the real axis onto itself and the imaginary axis onto the complement of this cut on the real axis,

$$q_S(\mathrm{i}\mathbb{R}) = \left(-\infty, -\frac{\omega}{\sqrt{\mu_0}}\right] \cup \left[\frac{\omega}{\sqrt{\mu_0}}, \infty\right). \tag{A2}$$

Furthermore,

$$q_S(\mathrm{i}\mathbb{R}_\pm) = \mathbb{R}_\mp\setminus\left(-\frac{\omega}{\sqrt{\mu_0}}, \frac{\omega}{\sqrt{\mu_0}}\right), \quad q_S\left(\mathbb{R}_\pm\setminus\left(-\frac{\omega}{\sqrt{\mu_0}}, \frac{\omega}{\sqrt{\mu_0}}\right)\right) = \mathrm{i}\mathbb{R}_\pm,$$

and

$$\pm\operatorname{Im}(q_S(\xi)) > 0 \ \text{iff} \ \xi \in \mathscr{K}_{S,\pm} = \{\xi \in \mathscr{K}_S \ : \ \pm\operatorname{Re}\xi > 0\}.$$

The Riemann surface for $q_S(\xi)$ is obtained by joining the upper and lower rims of two copies of $\mathbb{C}\setminus[(-\infty, -\frac{\omega}{\sqrt{\mu_0}}] \cup [\frac{\omega}{\sqrt{\mu_0}}, \infty)]$ cut along the $(-\infty, -\frac{\omega}{\sqrt{\mu_0}}] \cup [\frac{\omega}{\sqrt{\mu_0}}, \infty)$ in the usual (cross-wise) way. Instead of this two-sheeted Riemann surface, it is more convenient to work on the cut plane $\mathscr{K}_S$ and half planes $\mathscr{K}_{S,\pm}$ such that $q_S(\mathscr{K}_{S,\pm}) = \mathbb{C}_\pm := \{z \in \mathbb{C} \ : \ \pm\operatorname{Im} z > 0\}$. The "upper" (physical) sheet for $q_S$ corresponds to $\mathscr{K}_{S,+}$.

We collect below some useful properties,

$$\begin{aligned}
&\operatorname{Im} q_S(\xi) > 0 \ \text{iff} \ \xi \in \mathscr{K}_{S,+},\\
\text{for } \xi \in \mathbb{C}\setminus\left(\left[-\frac{\omega}{\sqrt{\mu_0}}, \frac{\omega}{\sqrt{\mu_0}}\right] \cup \mathrm{i}\mathbb{R}\right):& \quad q_S(\xi) = -q_S(-\xi) = -\overline{q_S(\bar{\xi})},\\
\text{for } \xi \in \left[-\frac{\omega}{\sqrt{\mu_0}}, \frac{\omega}{\sqrt{\mu_0}}\right]:& \quad q_S(\xi \pm \mathrm{i}\,0) = \mp\left|\frac{\omega^2}{\mu_0} - \xi^2\right|^{1/2},\\
\text{for } \xi \in \mathrm{i}\mathbb{R},:& \quad q_S(\xi \pm 0) = \mp\left|\frac{\omega^2}{\mu_0} - \xi^2\right|^{1/2},\\
\text{for } \xi \in \left(-\infty, -\frac{\omega}{\sqrt{\mu_0}}\right] \cup \left[\frac{\omega}{\sqrt{\mu_0}}, \infty\right):& \quad q_S(\xi) = \pm\mathrm{i}\left|\xi^2 - \frac{\omega^2}{\mu_0}\right|^{1/2}, \quad \pm\xi \geq \frac{\omega}{\sqrt{\mu_0}}.
\end{aligned} \tag{A3}$$

By replacing $\mu_0$ with $\sigma_0 := \lambda_0 + 2\mu_0$, we get analogous properties for quasimomentum,

$$q_P(\xi) = \sqrt{\frac{\omega^2}{\sigma_0} - \xi^2}.$$

Corresponding objects get the subscript $P$ instead of $S$. We introduce

$$\mathscr{K}_P := \mathbb{C}\backslash\left(\left[-\frac{\omega}{\sqrt{\sigma_0}}, \frac{\omega}{\sqrt{\sigma_0}}\right] \cup \mathrm{i}\mathbb{R}\right).$$

We observe that $q_P: \ \xi \to \sqrt{\frac{\omega^2}{\sigma_0} - \xi^2}$ is a conformal mapping $\mathscr{K}_P \to \mathscr{K}_P$. We obtain the Riemann surface $\mathscr{R}$ for both $q_P$ and $q_S$ by joining the separate Riemann surfaces for $q_P$ and $q_S$ so that $q_P$ and $q_S$ are single-valued holomorphic functions of $\xi$. We note that $\mathscr{R}$ is a four-fold cover of the plane. We identify the part of $\mathscr{R}$ where $\operatorname{Im} q_P > 0$, $\operatorname{Im} q_S > 0$ with the physical ("upper") sheet, which coincides with $\mathscr{K}_{S,+}$. Each sheet can be identified by the signs of $\operatorname{Im} q_S$ and $\operatorname{Im} q_P$. We omit the subscript $S$ in the notation and write $\mathscr{K} = \mathscr{K}_S$ and $\mathscr{K}_+ = \mathscr{K}_{S,+}$ for the cut plane and the part of the cut plane corresponding to the physical sheet, respectively. We note that $\xi$ has the meaning of Regge parameter.

In the main text, we introduce $\zeta = \xi^2$. We note that $\operatorname{Im} q_S(\zeta) > 0$, $\operatorname{Im} q_P(\zeta) > 0$ for $\zeta \in \Pi_+$, where

$$\Pi_+ = \mathbb{C}\backslash\left(-\infty, \frac{\omega^2}{\mu_0}\right] \tag{A4}$$

corresponds to the physical sheet, while $(\mathscr{K}_+)^2 = \Pi_+$. We introduce the notation

$$\Pi_{+,1} = \overline{\Pi_+}\backslash\left\{\frac{\omega^2}{\mu_0}\right\}, \quad \mathscr{K}_{+,1} = \overline{\mathscr{K}_+}\backslash\left\{\frac{\omega}{\sqrt{\mu_0}}\right\}.$$

We will use both parameters $\xi$ (Jost solutions and Jost function) and $\zeta$ (Weyl solutions and Weyl matrix) and both cut planes $\mathscr{K}_+$ and $\Pi_+$, switching between them when it appears natural.

## APPENDIX B: GREEN'S FUNCTION

We have

$$\mathfrak{G}(x,y) = [F^{(1)}(x,y) \ F^{(2)}(x,y)],$$

where $F^{(1)}(x,y)$, $F^{(2)}(x,y)$ are solutions of (39) for $x > y$, with boundary values

$$F^{(1)}(y,y) = \begin{pmatrix}0\\0\end{pmatrix}, \quad (F^{(1)})'(y,y) = \begin{pmatrix}1\\0\end{pmatrix}, \quad F^{(2)}(y,y) = \begin{pmatrix}0\\0\end{pmatrix}, \quad (F^{(2)})'(y,y) = \begin{pmatrix}0\\1\end{pmatrix}.$$

By explicit construction, we obtain the following.

*Lemma B.1. The following holds true:*

$$\mathfrak{G}(x,y) = \mathfrak{A}(x)\frac{\sin((x-y)q_P)}{q_P} + \mathfrak{B}(y)\frac{\sin((x-y)q_S)}{q_S} + \mathfrak{C}\frac{\cos((x-y)q_S) - \cos((x-y)q_P)}{\omega^2}, \tag{B1}$$

*where*

$$\mathfrak{A}(x) = \begin{pmatrix} G_{12}^H\left(\frac{c_0}{2}G_{11}^H(x-H) - G_{21}^H\right) & G_{11}^H\left(-\frac{c_0}{2}G_{11}^H(x-H) + G_{21}^H\right) \\ G_{12}^H\left(\frac{c_0}{2}G_{12}^H(x-H) - G_{22}^H\right) & G_{11}^H\left(-\frac{c_0}{2}G_{12}^H(x-H) + G_{22}^H\right) \end{pmatrix},$$

$$\mathfrak{B}(y) = \begin{pmatrix} G_{11}^H\left(\frac{c_0}{2}(-y+H)G_{12}^H + G_{22}^H\right) & -G_{11}^H\left(\frac{c_0}{2}(-y+H)G_{11}^H + G_{21}^H\right) \\ G_{12}^H\left(\frac{c_0}{2}(-y+H)G_{12}^H + G_{22}^H\right) & -G_{12}^H\left(\frac{c_0}{2}(-y+H)G_{11}^H + G_{21}^H\right) \end{pmatrix},$$

$$\mathfrak{C} = \begin{pmatrix} \mu_0 G_{12}^H G_{11}^H & -\mu_0 (G_{11}^H)^2 \\ \mu_0 (G_{12}^H)^2 & -\mu_0 G_{12}^H G_{11}^H \end{pmatrix}$$

*[cf. (32)].*

We note that $\mathfrak{A}(x)$ and $\mathfrak{B}(y)$ are first-order matrix-valued polynomials in $x$ and $y$, respectively, while $\mathfrak{C}$ is a constant matrix.

### 1. Homogeneous half space

In a homogeneous half space when $H = 0$, with $\mu = \mu_0$ and $G^H_{12} = G^H_{21} = 0$, $G^H_{11} = G^H_{22} = 1$, (B1) reduces to

$$\mathfrak{G}(x,y) = \begin{pmatrix} \frac{c_0}{2}(-y)\frac{\sin((x-y)q_S)}{q_S} & -\frac{c_0}{2}\left[x\frac{\sin((x-y)q_P)}{q_P} - y\frac{\sin((x-y)q_S)}{q_S}\right] + \mu_0\left[\frac{\cos((x-y)q_P)-\cos((x-y)q_S)}{\omega^2}\right] \\ 0 & \frac{\sin((x-y)q_P)}{q_P} \end{pmatrix}. \tag{B2}$$

## APPENDIX C: WEYL MATRIX AND NEUMANN-TO-DIRICHLET MAP OF THE RAYLEIGH SYSTEM

In this appendix, we study relationships between the orginal and transformed problems, that is, the Jost and Weyl solutions and the Jost function before the Markushevich transform. Consistent with the notation in (13), we let

$$\mathbf{w}(x,\xi) = \mathfrak{M}^{-1}(\mathbf{F})(x,\xi), \tag{C1}$$

where $\mathbf{F}$ signifies the Jost solution [cf. (52)], and we write

$$\mathbf{w} = [w_P\ w_S] \quad \text{and} \quad \widetilde{w}^- = [\widetilde{w}_P^-\ \widetilde{w}_S^-] \tag{C2}$$

[cf. (6)], supplemented with boundary conditions (9) and (10),

$$\mathbf{B}(\mathbf{w}) = \begin{pmatrix} b_-(w_P) & b_-(w_S) \\ a_-(w_P) & a_-(w_S) \end{pmatrix} = \begin{pmatrix} 0 & \mathrm{i} \\ -1 & 0 \end{pmatrix}\begin{pmatrix} a_-(\widetilde{w}_P^-) & a_-(\widetilde{w}_S^-) \\ b_-(\widetilde{w}_P^-) & b_-(\widetilde{w}_S^-) \end{pmatrix}. \tag{C3}$$

In the right-most equality, we reverted to the original notation [cf. (6)]. Setting $\chi = \mathbf{B}(\mathbf{w})$, (38) yields the expression for the Jost function,

$$\mathbf{F}_\Theta(\xi) = (D^{\mathrm{a}}(\xi))^{-1}\mathbf{B}(\mathbf{w}). \tag{C4}$$

In a likewise manner, we obtain for the adjoint problem,

$$\mathbf{F}^{\mathrm{a}}_\Theta(\xi) = (D(\xi))^{-1}\mathbf{B}(\mathbf{w}). \tag{C5}$$

Substituting (C3) into (C4) then gives

$$\mathbf{F}_\Theta(\xi) = \frac{1}{2\mu_0\mu(0)\xi}\begin{pmatrix} \mu(0) & 0 \\ 2\mu_0\frac{\mu'(0)}{\mu(0)} & 2\mu_0\xi\mathrm{i} \end{pmatrix}\begin{pmatrix} a_-(\widetilde{w}_P^-) & a_-(\widetilde{w}_S^-) \\ b_-(\widetilde{w}_P^-) & b_-(\widetilde{w}_S^-) \end{pmatrix}. \tag{C6}$$

Substituting (C3) into (C5) gives

$$\mathbf{F}^{\mathrm{a}}_\Theta(\xi) = \frac{1}{2\mu_0\mu(0)\xi}\begin{pmatrix} -2\mu_0\xi & 0 \\ 0 & -\mu(0)\mathrm{i} \end{pmatrix}\begin{pmatrix} a_-(\widetilde{w}_P^-) & a_-(\widetilde{w}_S^-) \\ b_-(\widetilde{w}_P^-) & b_-(\widetilde{w}_S^-) \end{pmatrix}. \tag{C7}$$

We subject the Weyl solution to $\mathfrak{M}^{-1}$ [cf. (14)], substitute (70), and introduce

$$\boldsymbol{r}(x,\xi) = \mathfrak{M}^{-1}(\boldsymbol{\Phi})(x,\xi) = \boldsymbol{\theta}(x,\xi) + \boldsymbol{\psi}(x,\xi)\mathbf{M}(\xi), \tag{C8}$$

with

$$\boldsymbol{\theta}(x,\xi) = \mathfrak{M}^{-1}(\mathbf{S})(x,\xi), \quad \boldsymbol{\psi}(x,\xi) = \mathfrak{M}^{-1}(\boldsymbol{\varphi})(x,\xi). \tag{C9}$$

Using the definition of Weyl solution, we find that

$$\mathbf{w}(x,\xi) = \boldsymbol{r}(x,\xi)\mathbf{F}_\Theta(\xi), \tag{C10}$$

where we write

$$\boldsymbol{r} = [r_P\ r_S]. \tag{C11}$$

Equation (C4) implies that

$$\mathbf{B}(\boldsymbol{r}) = \begin{pmatrix} b_-(r_P) & b_-(r_S) \\ a_-(r_P) & a_-(r_S) \end{pmatrix} = \chi_{\mathrm{I}} = D^{\mathrm{a}}(\xi),$$

where we used (C3). Substituting (C4) into (C10), we get

$$\mathbf{w}(x,\xi) = \boldsymbol{r}(x,\xi)(D^{a}(\xi))^{-1}\mathbf{B}(\mathbf{w}). \tag{C12}$$

Now, recalling the relation between a solution to system (7)-(8) [cf. (C11)],

$$\boldsymbol{r}_{\bullet}(x,\xi) = \begin{pmatrix} r_{\bullet,1}(x,\xi) \\ r_{\bullet,2}(x,\xi) \end{pmatrix},$$

and a solution to system (2)-(3),

$$\tilde{\boldsymbol{r}}_{\bullet}(x,\xi) = \begin{pmatrix} \mathrm{i} r_{\bullet,1}(-Z,\xi) \\ r_{\bullet,2}(-Z,\xi) \end{pmatrix},$$

where • stands for either $P$ or $S$, the Neumann-to-Dirichlet map for the Rayleigh problem is given by

$$\begin{aligned}\mathbf{ND}(\xi) &= \tilde{\boldsymbol{r}}(0,\xi)\,(D^{a}(\xi))^{-1}\begin{pmatrix} 0 & \mathrm{i} \\ -1 & 0 \end{pmatrix} = (\tilde{\boldsymbol{\theta}}(0,\xi) + \tilde{\boldsymbol{\psi}}(0,\xi)\mathbf{M}(\xi))(D^{a}(\xi))^{-1}\begin{pmatrix} 0 & \mathrm{i} \\ -1 & 0 \end{pmatrix} \\ &= \left[\begin{pmatrix} \mathrm{i}\dfrac{\mu_0}{\mu(0)} & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \dfrac{1}{2}\mathrm{i} \\ -\dfrac{\mu_0}{\mu(0)}\xi & 0 \end{pmatrix}\mathbf{M}(\xi)\right]\begin{pmatrix} \dfrac{1}{2\mu_0\xi} & 0 \\ \dfrac{\mu'(0)}{\mu^2(0)}\dfrac{1}{\xi} & \dfrac{\mathrm{i}}{\mu(0)} \end{pmatrix}\end{aligned} \tag{C13}$$

[cf. (C8)]. This equation provides a direct relationship between the Weyl matrix and the (observable) Neumann-to-Dirichlet map and, more specifically, between the associated spectral data as $\boldsymbol{\theta}$ and $\boldsymbol{\psi}$ are entire functions in $\xi$. Substituting (61) into the equation above yields the Neumann-to-Dirichlet map in a homogeneous half space.

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