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Recovery of discontinuous Lamé parameters from exterior Cauchy data

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ABSTRACT

Consider an isotropic elastic medium $\Omega \subset \mathbb{R}^3$ whose Lamé parameters are piecewise smooth. In the elastic wave initial value inverse problem, we are given the solution operator for the elastic wave equation, but only outside Ω and only for initial data supported outside Ω . Using the recently introduced scattering control series in the acoustic case, we prove that piecewise smooth Lamé parameters are uniquely determined by this map under certain geometric conditions.

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1. Introduction

The wave inverse problem asks for the unknown coefficient(s), representing wave speeds, of a wave equation inside a domain of interest Ω , given knowledge about the equation's solutions (typically on $\partial\Omega$). Traditionally, the coefficients are smooth, and the data is the Dirichlet-to-Neumann (DN) map, or its inverse. The main questions are uniqueness and stability: can the coefficients be recovered from the Dirichlet-to-Neumann map, and is this reconstruction stable relative to perturbations in the data? In the case of a scalar wave equation with smooth coefficients, a number of results by Belishev, Stefanov, Vasy, and Uhlmann [1–3] have answered the question in the affirmative. For the piecewise smooth case, a novel scattering control method was developed in [4] in order to show in [5] that uniqueness holds as well for piecewise smooth wave speeds with conormal singularities, under very mild geometric conditions. We term that particular method as *blind scattering control* since it assumes absolutely no knowledge of the wave speed in the interior region, and uses only measurements exterior to Ω . Our goal is to extend these results to the isotropic elastic system. This presents new difficulties due to the lack of the sharp form of the unique continuation result of Tataru since we have to deal with two different wave speeds.

In the elastic setting, or for that matter, any hyperbolic equation with multiple wave speeds, the story is far from complete. Consider the isotropic elastic wave equation in a

bounded domain Ω with smooth boundary. The wave operator for elastodynamics is given as $Q = \rho \partial_t^2 - L$ with

$$L = \nabla \cdot (\lambda \operatorname{div} \otimes \operatorname{Id} + 2\mu \widehat{\nabla}),$$

ρ is the density, λ and μ are the Lamé parameters, and $\widehat{\nabla}$ is the *symmetric gradient* used to define the strain tensor for an elastic system via $\widehat{\nabla}u = (\nabla u + (\nabla u)^T)/2$ for a vector valued function u . Operator Q acts on a vector-valued distribution $u(x, t) = (u_1, u_2, u_3)$, the *displacement* of the elastic object. For the isotropic, elastic setting with smooth parameters, the uniqueness question was settled by Rachele in [6] and Hansen and Uhlmann [7]. First, Rachele proved that one can recover the jet of λ , μ , and ρ at $\partial\Omega$ explicitly. In [6, 8], she showed that one can recover the P and S wave speeds in Ω provided the hyperbolic DN map is known on the whole boundary and assuming strict geometry that preclude caustics. Hansen and Uhlmann studied the problem with a residual stress, allowing conjugate points and caustics, and showed that one can recover both lens relations and derived the consequences of that. These are all results for the global problem where the DN map is known on the whole boundary. Stefanov, Vasy, and Uhlmann [3, 9] have extended these results to the local inverse problem using the Uhlmann-Vasy methods on the local geodesic ray transform [3] and using a pseudolinearization first developed in [2]. They are able to do a local recovery of both wave speeds that depend on three parameters λ, μ, ρ . There are also related inverse problems in *thermoacoustic tomography* where one tries to recover a source (initial condition) rather than a PDE parameter [10–14].

No such results are known for when the elastic parameters have interfaces (conormal singularities). Even the blind scattering control method, which is very similar to boundary control and was used to prove uniqueness in [5] for the acoustic setting, does not readily apply here. The reason is very simple: although unique continuation results hold for the elastic setting, they are far weaker, being based on the slowest wave speed, and so the boundary control method is not known to work since it is not possible, or at least not known, how to decouple the elastic system completely even though it is easy to do that microlocally. In [15], the authors define a Marchenko-type algorithm via a Neumann series to eliminate and control multiple scattering in the elastic setting. However, it is not mathematically rigorous and they assume knowledge of first arrival times corresponding to purely transmitted P waves, S waves, and certain mode converted waves. This is a strong assumption since a single wave packet entering Ω produces numerous scattered waves that one measures at the surface and one cannot *a priori* associate travel times with a particular primary reflected wave versus a secondary reflected wave.

For another approach, a Lamé type of system having the same principal part which can be decoupled fully was studied by Belishev in [16] and the boundary control method (see [1, 17, 18]) was used for unique recovery. Such an approach only worked because the system was able to fully decouple so that the scalar boundary control methods would apply to the decoupled constituents. Therefore, it fails for the piecewise smooth setting where the coupling of different modes at the interfaces is unavoidable, and so it does not simplify matters here to study Belishev's Lamé type system with

piecewise smooth parameters. Instead, we focus on a geometric uniqueness problem analogous to [9] and employ a layer stripping argument to utilize the results in [3, 9] in the smooth case. Our proof is based on techniques from microlocal analysis. While this paper was under review, we became aware of another submission [19] where the authors carefully construct solutions microlocally of the transmission problem at a nested set of interfaces, and apply their analysis to the conditional recovery of P - and S -wave speeds in a layer-wise way.

The main result of this paper is that under certain geometric assumptions, we show unique determination of Lamé parameters that contain singularities via microlocal analysis, scattering control, and a layer stripping argument akin to [9]. Most proofs are microlocal to avoid using unique continuation results, but we require an important geometric assumption, which is an *extended convex foliation condition* (see §3 for the smooth setting) for each wave speed $c_{P/S}$. As mentioned in [9], for a particular wave speed, this condition relates to the existence of a function with strictly convex level set. In particular, this holds for simply connected compact manifolds with strictly convex boundaries such that the geodesic flow has no focal points (lengths of non-trivial Jacobi fields vanishing at a point do not have critical points), in particular if the curvature of the manifold is negative (or just non-positive). Also, as explained in [20], if Ω is a ball and the speeds increase when the distance to the center decreases (typical for geophysical applications), the foliation condition is satisfied.

The other key ingredient is that even though a lens map does not make sense with internal multiples present, one may use a scattering control-like process introduced in [4] to recover lens data for singly reflected rays. This construction will also be entirely microlocal and circumvents the need for unique continuation results. We denote by u_h to be the solution to the homogeneous elastic equation on \mathbb{R}^3 with initial time Cauchy data h . All of our function spaces are of the form $X(\cdot; \mathbb{C}^3)$ since we have vector valued functions in the elastic setting, but throughout the paper, we will not write the vector valued part \mathbb{C}^3 to make the notation less burdensome. Let $\bar{\Omega}^c$ be the complement of $\bar{\Omega}$ and we define the *exterior measurement operator* $\mathcal{F} : H_c^1(\bar{\Omega}^c) \oplus L_c^2(\bar{\Omega}^c) \rightarrow C^0(\mathbb{R}_t; H^1(\bar{\Omega}^c)) \cap C^1(\mathbb{R}_t; L^2(\bar{\Omega}^c))$ as

$$\mathcal{F} : h_0 \rightarrow u_{h_0}(t)|_{\bar{\Omega}^c}.$$

Due to a technicality, we use slightly different sets for our measurement region than $\bar{\Omega}^c$ in the main body, but the idea is the same. The operator \mathcal{F} only measures waves outside Ω after undergoing scattering within Ω , and it is associated to a particular elastic operator Q with a set of parameters. Given a second set of elastic parameters $\tilde{\lambda}, \tilde{\mu}$ we obtain analogous operators \tilde{Q} and $\tilde{\mathcal{F}}$. Denote the associated P/S wave speeds $c_{P/S}$ and $\tilde{c}_{P/S}$. From here on, we use P/S to refer to either subscript or wave speed. In addition, to avoid the technical difficulties of dealing with corners or higher codimension singularities of $c_{P/S}$, we always assume that the singular support of $c_{P/S}, \tilde{c}_{P/S}$ lies in a closed, not necessarily connected hypersurface in Ω ; we will deal with corners and edges in a separate paper.

We assume the Lamé parameters $\lambda(x)$ and $\mu(x)$ satisfy the *strong convexity condition*, namely that $\mu > 0$ and $3\lambda + 2\mu > 0$ on $\bar{\Omega}$. We also assume that the parameters λ, μ lie in $L^\infty(\Omega)$ and that λ, μ are piecewise smooth functions that are singular only on a set of

disjoint, closed, connected, smooth hypersurfaces Γ_i of $\tilde{\Omega}$, called *interfaces*. We let $\Gamma = \cup \Gamma_i$ be the collection of all the interfaces. The two wave speeds are $c_P = \sqrt{(\lambda + 2\mu)/\rho}$ and $c_S = \sqrt{\mu/\rho}$, where ρ is the density. In particular, this ensures that $c_P > c_S$ on $\tilde{\Omega}$.

As in [4], we will probe Ω with Cauchy data (an *initial pulse*) concentrated close to Ω with a particular polarization, in some smooth domain $\Theta \supset \Omega$. Since we take measurements outside Ω , let us extend the Lamé parameters to all of \mathbb{R}^n so that they are smooth outside $\tilde{\Omega}$ and our wavefields are now well-defined there as well. We will denote by

$$g_{P/S} = c_{P/S}^{-2} dx^2$$

the two different metrics associated to the rays. As in [4], we can define the distance functions $d_{P/S}(\cdot, \cdot)$ corresponding to the respective metrics by taking the infimum over all lengths of the piecewise smooth paths between a pair of points. Here and throughout the paper, a P/S subscript indicates either P or S subscripts.

Now, define the P -depth $d_{\Theta}^*(x)$ of a point x inside Θ :

$$d_{\Theta}^*(x) = \begin{cases} +d_P(x, \partial\Theta), & x \in \Theta, \\ -d_P(x, \partial\Theta), & x \notin \Theta. \end{cases}$$

We use the (rough) metric g_P since finite speed of propagation for elastic waves is based on the faster P -wave speed. We will prove the following result.

Theorem 1.1. *Assume $\mathcal{F} = \tilde{\mathcal{F}}$, and that $c_{P/S}, \tilde{c}_{P/S}$ satisfy the extended geometric foliation condition (see Section 3). Then $c_P = \tilde{c}_P$ and $c_S = \tilde{c}_S$ inside Ω .*

Via a layer stripping approach, we will obtain local travel time data and lens relations at the current layer from \mathcal{F}^1 . To do this, we will employ an analogue of the microlocal scattering control construction appearing in [4, section 5] to create specific P or S waves at the current, deepest layer to extract local travel time data and lens relations without having the internal multiples interfere with recovery of this data. Without such techniques, one would not be able to distinguish waves that contain this subsurface travel time data from internal multiples created from the conormal singularities of the Lamé parameters.

Remark 1.2. We note that a large portion of the proof is in principle constructive. In [Appendix B](#) we use the calculus of Fourier integral operators (FIOs) to explicitly construct initial sources that allow us to extract local travel time data in the interior. As we layer strip, we progressively obtain new information in the interior in order to make such constructions. Using local travel time data, the authors in [3, 9] construct the normal operator of the local ray transform that allows local reconstruction of a first order perturbation of the wave speed. It should be possible to generate a reconstruction algorithm in a future work using some of our constructions here and those in [3].

¹The fact that the interfaces are not dense makes this possible theoretically in the sense that there will exist an open set of rays at the current layer that *do not cross any interfaces* after a finite time when they are close to being tangent to the layer.

Remark 1.3. We note that our arguments here are microlocal while in a previous paper [5] for the scalar wave equation, we used exact constructions that allowed for a full reconstruction algorithm as well. We did not need the extended convex foliation condition in the acoustic-wave case [5] as we could employ strong unique continuation. We could follow a similar strategy in the elastic-wave case for the slow (S-wave) domain of influence only. However, this would only allow partial elimination of scattering in the elastic case (based on the slow domain of influence) and not the full control necessary for this problem. Thus, we avoid unique continuation altogether by working with microlocal solutions and studying wavefront sets.

Index of notation

Due to numerous new notation in this manuscript, we provide a brief index on the symbols we use with the page number they are introduced. If a symbol is used infrequently, we list the other pages it appears.

Symbol	Summary	Pages
Ω	The domain representing the elastic object	1
Q	The elastic wave operator	1
\mathcal{F}	The exterior measurement operator	2
$c_{P/S}$	The P- and S- wave speeds	3
Γ_i	The i 'th interface	3
Γ	Collection of all the interfaces	3
Θ	Small neighborhood of Ω	3
$g_{P/S}$	Metric corresponding to the P/S wave speeds	3
$d_{\Theta}^P(x)$	The depth function corresponding to the P wave speed	3
F	Solution operator to the elastic wave initial value problem	6
R_s	Propagator operator that propagates Cauchy data by s units of time.	6
ν	Time reversal operator that acts on Cauchy data	6
$\Sigma_{P/S}$	The P/S characteristic sets	6
ρ	Convex foliation function	7
$L_{P/S,\tau}(x, \xi)$	subsurface lens relation within Ω_τ	7
Ω_τ	Subdomains of Ω determined by the foliation function ρ	8
Σ_τ	A leaf of the foliation; that is $\rho^{-1}(\tau)$	8
$T_\pm^i \Omega$	Set of foliation upward/downward covectors	8,10
$l_{P/S,\tau}(x, \xi)$	subsurface travel time used to measure boundary distances on Ω_τ	9
Z	A disjoint union of connected components of $\mathbb{R}^3 \setminus \Gamma$ so that ∂Z contains two copies of the interface	10
Z	A doubled space of Z to use for the two components of Cauchy data.	10
$J_{C \rightarrow S}$	Cauchy to solution operator	10
$J_{C \rightarrow S+}$	forward Cauchy to solution operator	10
$J_{C \rightarrow \partial}$	Cauchy to boundary map	10
$J_{C \rightarrow \partial+}$	forward Cauchy to boundary map	10
$J_{\partial \rightarrow S}$	boundary to solution map	11
$\Pi_{P/S}$	Microlocal projectors of an elastic wavefield u onto the P/S-characteristic set	11
$J_{\partial \rightarrow \partial}$	boundary to boundary propagator map; propagates boundary data to the next boundary that the waves intersect	11
$M_{R/T}$	reflection and transmission operators	12
S	Set of covectors in $T^* \Omega$ such that every unutilized bicharacteristic belonging to a broken bicharacteristic through the covector is (+)-escapable	14
$d_{P/S}^+(\cdot, \cdot)$	P/S distance function restricted to $\Omega_\tau \times \Omega_\tau$	19
$DT_{k,p}^+$	directly transmitted constituent of the wavefield	18, 29

2. Preliminary setup

We will use this section to give the basic definitions and setup for the main theorem.

Recall from the introduction that Ω is a bounded region in \mathbb{R}^3 with smooth boundary. It represents a linearly elastic, inhomogeneous, isotropic object. We will add to the initial pulse a Cauchy data control (a *tail*) supported outside Θ , whose role is to remove multiple reflections up to a certain depth, controlled by a time parameter $T \in (0, \frac{1}{2} \text{diam}_p \Omega)$. This will require us to consider controls supported in a sufficiently large Lipschitz neighborhood $\Upsilon \subsetneq \mathbb{R}^3$ of $\bar{\Theta}$ that satisfies $d_S(\partial \Upsilon, \bar{\Theta}) > 2T$ and is otherwise arbitrary. It will be useful to define $\Theta^* = \{x \in \Upsilon \mid d_{\Theta}^*(x) < 0\}$.

2.1. Elastic waves

Recall that the wave operator for elastodynamics Q discussed in the introduction is $Q = \rho \partial_t^2 - L$ with

$$L = \nabla \cdot (\lambda \text{div} \otimes \text{Id} + 2\mu \widehat{\nabla}).$$

Let us also recall the characteristic set of Q defined in [6] and [7]. It consists of two mutually disjoint sets $\Sigma_P, \Sigma_S \subset T^*\mathbb{R}^3$ where $\Sigma_{P/S}$ are the characteristic sets for the scalar wave operators $c_{P/S}^{-2} \partial_t^2 - \Delta$.

Let $\tilde{\mathbf{C}}$ be the space of Cauchy data of interest:

$$\tilde{\mathbf{C}} = H_0^1(\Upsilon; \mathbb{C}^3) \oplus L^2(\Upsilon; \mathbb{C}^3)$$

although we will suppress the “ \mathbb{C}^3 ” notation when it is clear from the context. We equip the space with the *elastic energy inner product*

$$\langle (f_0, f_1), (g_0, g_1) \rangle = \int_{\Omega} \left(f_1 \cdot \bar{g}_1 + \lambda(x) \text{div}(f_0) \text{div}(\bar{g}_0) + 2\mu(x) \widehat{\nabla} f_0 : \widehat{\nabla} \bar{g}_0 \right) dx.$$

Within $\tilde{\mathbf{C}}$, define the subspaces of Cauchy data supported inside and outside Θ_t :

$$\mathbf{H} = H_0^1(\Theta) \oplus L^2(\Theta), \quad \tilde{\mathbf{H}}^* = H_0^1(\Theta^*) \oplus L^2(\Theta^*).$$

Define the energy of Cauchy data $h = (h_0, h_1) \in \tilde{\mathbf{C}}$ in a subset $W \subset \mathbb{R}^3$:

$$\mathbf{E}_W(h) := \int_W \left(\lambda(x) |\text{div}(h_0)|^2 + \mu(x) |\widehat{\nabla} h_0|^2 + |h_1|^2 \right) dx.$$

Next, define F to be the solution operator for the elastic wave initial value problem:

$$F : H^1(\mathbb{R}^n) \oplus L^2(\mathbb{R}^3) \rightarrow C(\mathbb{R}, H^1(\mathbb{R}^3)) \quad F(h_0, h_1) = u \text{ s.t. } \begin{cases} \mathbf{Q}u & = 0, \\ u|_{t=0} & = h_0, \\ \partial_t u|_{t=0} & = h_1. \end{cases} \quad (2.1)$$

Let R_s propagate Cauchy data at time $t=0$ to Cauchy data at $t=s$:

$$R_s = (F, \partial_t F)|_{t=s} : H^1(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3) \rightarrow H^1(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3). \quad (2.2)$$

Now combine R_s , with a time-reversal operator $\nu : \tilde{\mathbf{C}} \rightarrow \tilde{\mathbf{C}}$, defining for a given T

$$R = \nu \circ R_{2T}, \quad \nu : (f_0, f_1) \mapsto (f_0, -f_1).$$

In our problem, only waves interacting with (Ω, μ, λ) in the time interval $[0, 2T]$ are of interest. Despite not having full unique continuation, one is still able to construct parametrices for the elastic equation that are approximate solutions to the elastic equation. With such parametrices, we will use the principles from scattering control to obtain “subsurface” travel time data by eliminating certain scattered constituents micro-locally. This is precisely the task we pursue in the remainder of the paper. First, we develop our main technical tool, which is the parametrix for elastic wave solutions.

3. Foliation condition

The main theorem is showing that both the P and S wave speeds are determined by the outside-measurement-operator under certain geometric conditions that give us access to all the requisite rays. We use this section to describe these geometric conditions.

First, since our proof of the main theorem will require recovery of all the parameters in a layer stripping argument, we make a simplifying assumption and assume the density

$$\rho = 1$$

throughout the paper.

Remark 3.1. We needed to assume the density $\rho = 1$ for the proof of the main theorem in order to recover all Lamé parameters during the layer stripping procedure, thereby giving us access to the full wave solution in the known layers. In [21], Rachele shows how one may use “lower order polarization” data to recover the density ρ as well under certain conditions. However, this was in the smooth setting and the result was global since it utilized a global inversion result of an X-ray transform of tensor fields. Since that paper, Stefanov, Uhlmann, and Vasy in [22] have shown that one may also obtain local inversion results of the X-ray transform on tensors. Hence, it may be possible to combine Rachele’s argument to obtain local, lower order polarization data containing information on the density from the outside measurement operator combined with the result in [22] on the local ray transform on tensors to recover the density ρ during our layer stripping procedure. We will pursue this strategy in another work.

Let us recall all the definitions from [5], adapted to the elastic setting. We start by extending the convex foliation condition to our piecewise smooth setting, keeping in mind that Γ_i, Γ are the interfaces defined in section 2.1.

Definition 3.2. $\rho : \bar{\Omega} \rightarrow [0, \tau_0]$ is a (piecewise) *extended convex foliation* for $(\Omega, c_{P/S})$ (meaning for both c_P and c_S simultaneously) if the following conditions hold:

- $\partial\Omega = \rho^{-1}(0)$ and $\rho^{-1}(\tau_0)$ has measure zero;
- ρ is smooth and $d\rho \neq 0$ on $\rho^{-1}((0, \tau_0)) \setminus \Gamma$.
- each level set $\rho^{-1}(t)$ is geodesically convex with respect to c_P and c_S when viewed from $\rho^{-1}((t, T))$, for $t \in [0, \tau_0)$.
- the interfaces of $c_{P/S}$ are level sets of ρ_i , that is $\Gamma_i \subset \rho^{-1}(t_i)$ for some t_i .
- ρ is upper semicontinuous.
- $\limsup_{\epsilon \rightarrow 0^+} c_{P/S}|_{\rho^{-1}(\tau+\epsilon)} \leq \limsup_{\epsilon \rightarrow 0^+} c_{P/S}|_{\rho^{-1}(\tau-\epsilon)}$ whenever $\Gamma_i \subset \rho^{-1}(\tau)$ for some i and Γ_i .

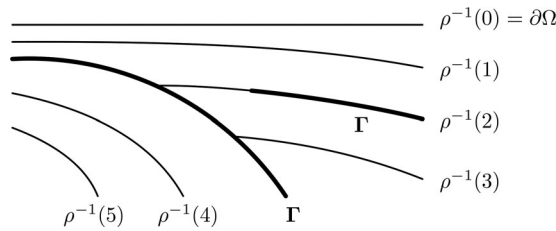


Figure 1. An example of a piecewise convex foliation. Thick lines indicate the interfaces Γ ; thin lines trace selected level sets of the foliation function ρ , which is allowed (but not required) to be singular at Γ .

We say that (c_p, c_s) satisfies the *extended convex foliation condition* if there exists a extended convex foliation for $(\Omega, c_{p/s})$.

The last condition, which did not appear in [2], is natural for the case of interfaces. It ensures that rays do not get trapped due to total internal reflection. This is the analog of the Herglotz condition extended to jump discontinuities as in [23, Definition 1]. Hence, a *P/S*-ray approaching Γ from “below” (defined below) will get transmitted above the interface. Also, any smooth approximation to $c_{p/s}$ that satisfies the other properties of the extended foliation condition would automatically satisfy this condition by definition of geodesic convexity. Other non-trivial examples can be constructed from those described in [9, 24] for the smooth setting. As mentioned in [9] for the smooth setting, it follows from the result of [24], that manifolds with no focal points satisfy the regular foliation condition. Manifolds satisfying the foliation condition are not necessarily simple. Thus, according to those results, one may take two (or more) manifolds without focal points that are each foliated by convex geodesic spheres with defining function ρ_1 and ρ_2 (say). One can then glue portions of each manifold together at the boundary to create an interface. For example, say $M_1 = \rho_1^{-1}([\tau_a, \tau_b])$ and $M_2 = \rho_2^{-1}([\tau_c, \tau_d])$ for some $\tau_{b/d} > \tau_{c/a} > 0$. After possibly a diffeomorphism, one may glue M_1 to M_2 along $\rho_1^{-1}(\tau_a)$ and $\rho_1^{-1}(\tau_d)$ so that these geodesic spheres become an interface. By perturbing the metric slightly if necessary, one can ensure that the final condition in the extended foliation is satisfied, that is, the wave speed jump across this boundary has the correct sign.

Having interfaces being part of the foliation allows for some unusual configurations. In addition, the leaves of the foliation may have intricate, non-trivial topologies and the geometry can be complicated as well, allowing conjugate points (see below Figure 1).

From now on we assume

Assumption 1. $(\Omega, c_{p/s})$ satisfy the extended convex foliation condition.

We note that the case where c_p and c_s have separate foliations does not add much more generality to the theorem (see Remark 3.3).

Remark 3.3. We are assuming that the level sets of one function ρ produces an extended convex foliation for both the c_p and c_s wave speed. One may wonder whether this is strictly necessary since we recover the wave speeds one at a time in the proof. Upon close examination of the main proof, it will be vital that the interfaces coincide with the leaves of the foliation so that we get the correct scattering behavior that ensures enough branches of a particular ray return outside Ω , which is the measurement region. Thus, we may allow c_p and c_s to have different foliations, but the foliations

must still coincide at and near each interface. Dealing with interfaces is the main novelty of the paper so allowing different foliations away from the interfaces does not present much novelty to our results.

Nevertheless, one may wonder whether the extended convex foliation may be weakened near the boundary since we assume that the interfaces are a positive distance from the boundary. This is plausible, but the argument to recover the parameters near the boundary would be different than the one considered here, and would be analogous to [8]. Since our main concern is dealing with interfaces, we do not pursue that argument here.

We note that the extended convex foliation condition gives us crucial information on the reflected and transmitted waves emitted when an incident wave hits an interface. Indeed, define

$$\begin{aligned} \Omega_\tau &= \rho^{-1}((\tau, \tau_0]), & \Sigma_\tau &= \rho^{-1}(\tau). \\ \Omega_\tau^* &= \rho^{-1}((0, \tau)), \end{aligned} \tag{3.1}$$

Also, let Σ_τ^\pm denote the two sides of the interface, where $(-)$ refers to the outside of Ω_τ (facing decreasing τ) and $(+)$ the inside. We also fix such notation for the remainder of the paper. We have the corresponding sets of P/S hyperbolic points $\mathcal{H}_{P/S}^\pm \subset T^*\Gamma_\pm$ (see [7, section 4] for the relevant definitions). The convexity guarantees that $\mathcal{H}_P^+ \subset \mathcal{H}_P^-$ with an analogous statement for the S hyperbolic set due to the last condition in the extended convex foliation definition so that rays do not become trapped due to total internal reflection. Thus, a P wave hitting Σ_τ from below must produce a transmitted P wave. In fact, it must produce a transmitted S wave as well since $c_P > c_S$. The same holds for an S wave hitting Γ from below, but a mode conversion in the transmitted wave does not necessarily occur since mode conversions only occurs up to a critical angle. Thus, there is no total internal reflection from below the interfaces.

First, we need several definitions taken from [5] extended to the elastic setting.

Definition 3.4. A foliation downward (resp. upward) covector (x, ξ) is one pointing in direction of increasing (resp. decreasing) ρ . Define $T_\pm^*\Omega$ to be the associated open sets:

$$T_\pm^*\Omega = \{(x, \xi) \in T^*\Omega \mid \pm \langle \xi, d\rho \rangle > 0\}.$$

Hence, we can speak of covectors (x, ξ) pointing upward/downward with respect to the foliation.

Definition 3.5. A (unit-speed) broken geodesic in $(\mathbb{R}^n, c_{P/S})$ is a continuous, piecewise smooth path $\gamma : \mathbb{R} \supset I \rightarrow \mathbb{R}^n$ such that each smooth piece is a unit-speed geodesic with respect to either g_P or g_S on $\mathbb{R}^n \setminus \Gamma$, intersecting the interfaces Γ at discrete set of times $t_i \in I$. Furthermore, at each t_i the intersection is transversal and Snell’s law for reflections and refraction of elastic waves is satisfied. A broken bicharacteristic is a path in $T^*\mathbb{R}^n$ of the form (γ, γ^b) , the flat operation taken with respect to g_P or g_S as appropriate. Note that a broken geodesic defined this way may contain both P and S geodesic segments. More precisely, a broken bicharacteristic (parameterized by a time variable) can be written as $\gamma : (t_0, t_1) \cup (t_1, t_2) \cup \dots \cup (t_{k-1}, t_k) \rightarrow T^*\mathbb{R}^n \setminus \Gamma$, which is a sequence of bicharacteristics connected by reflections and refractions obeying Snell’s law: for $i = 1, \dots, k - 1$,

$$\gamma(t_i^-), \gamma(t_i^+) \in T_\Gamma^*(\mathbb{R}^n), \quad (d\iota_\Gamma)^* \gamma(t_i^-) = (d\iota_\Gamma)^* \gamma(t_i^+), \quad (3.2)$$

where $\iota_\Gamma : \Gamma \hookrightarrow \mathbb{R}^n$ is the inclusion map and $\gamma(t_i^\mp) = \lim_{t \rightarrow t_i^\mp} \gamma(t)$. We always assume that γ intersects the interfaces transversely since for our parametrix construction, we assume that solutions have wave front set disjoint from bicharacteristics tangential to any of the interfaces. Each restriction $\gamma|_{(t_i, t_{i+1})}$ is a *P-bicharacteristic*, respectively *S-bicharacteristic* if it is a bicharacteristic for $\partial_t^2 - c_P \Delta$, respectively $\partial_t^2 - c_S \Delta$. We also call each such bicharacteristic a *branch* of γ ; we are sometimes more specific and write *P branch* or *S branch* if we want to specify the associated metric. For each i , note that $\gamma(t_i)$ projected to the base manifold is a point of Γ_{k_i} for some k_i . A branch $\gamma|_{(t_i, t_{i+1})}$ is *reflected* if the inner product of $\gamma'(t_i^+)$ and $\gamma'(t_i^-)$ (when projected to base space) with a normal vector to Γ_{k_i} have opposite signs. Otherwise, it is a *transmitted branch*. Say that $\gamma|_{(t_i, t_{i+1})}$ is a *mode converted branch* if it is a *P/S branch* and $\gamma|_{(t_{i-1}, t_i)}$ is a *S/P branch*.

A *purely transmitted P/S broken geodesic* (a concatenation of smooth *P* or *S* geodesics) is a unit-speed broken geodesic that consists of only *P/S transmitted branches*; that is, the inner products of $\gamma'(t_i^-)$ and $\gamma'(t_i^+)$ with the normal to Γ have identical signs at each t_i and they are all either *P geodesics* or *S geodesics*. A *purely transmitted P/S broken bicharacteristic* is then defined the same way using projection to base space.

Definition 3.6. Let $(x, \zeta) \in T_+^* \bar{\Omega} \setminus 0$, and $\tau = \rho(x)$. If there exists a purely transmitted bicharacteristic γ (with either only *P* or only *S* branches) and $\lim_{t \rightarrow 0^+} \gamma(t) = (x, \zeta)$, we define the *subsurface travel time* $l_{P/S, \tau}(x, \zeta)$ as the smallest $l > 0$ for which $\gamma(l) \in T_-^* \Omega \cap T_{\Sigma_\tau}^* \Omega$, and the (*subsurface*) *lens relation* $L_{P/S, \tau}(x, \zeta) = \gamma(l)$.

If $\mathcal{D}_{P/S}$ is the set of (x, ζ) for which such γ exists, extend $L_{P/S}$ to $(\overline{\mathcal{D}_{P/S}} \setminus 0) \setminus T^* \Omega|_{\Gamma_\pm}$ by continuity. On the interfaces $T^* \Omega|_{\Gamma_\pm}$, define $L_{P/S}$ by continuity from below.

Definition 3.7. Let $\Omega_r \subseteq \Omega$ be the set of *regular points*, where x is regular if it is regular with respect to both c_P and c_S , as defined in [5, Definition 3.2].

Essentially, $x \in \Omega_r$ means that there is a purely transmitted broken *P* and *S* geodesic that starts normal to $\partial\Omega$ and passes through x . We do not go into detail on the definition of Ω_r since due to the extended convex foliation assumption, it is a dense set in Ω , which is all that we use in our proofs:

Lemma 3.8. *If Ω is compact, then Ω_r is dense in Ω under the extended convex foliation assumption.*

The proof is by applying Lemma 3.3 in [5] to both *P* and *S* speeds.

Since the proof of the main theorem is microlocal, we must first construct a parametrix for the elastic operator when the Lamé parameters are piecewise smooth.

4. Elastic-wave parametrix with scattering

In this section, we construct the elastic wave parametrix in the presence of singularities in the Lamé parameters. Most constructions are taken directly from [4] used in the

acoustic setting and [8] in the elastic setting. Other references to construct such parametrices are [25] in an acoustic setting and [26] for a general systems setting.

Let us first recall the interfaces Γ_i with $\Gamma = \cup \Gamma_i$. These hypersurfaces separate $\mathbb{R}^3 \setminus \Gamma$ into disjoint components $\{\Omega_j\}$. We assume each smooth piece of λ and μ extends smoothly to \mathbb{R}^3 . In order to distinguish the sides of each hypersurface Γ_i , consider an *exploded space* Z in which the connected components of $\mathbb{R}^3 \setminus \Gamma$ are separate. It may be defined in terms of its closure, as a disjoint union

$$\bar{Z} = \sqcup_j \bar{\Omega}_j, \quad Z = \cup_j \Omega_j \subset \bar{Z}.$$

In this way, ∂Z contains two copies of each Γ_i , one for each adjoining Ω_j .

When restricting to a particular Ω_j , we may do a microlocal decomposition into the forward and backward propagators as in the acoustic case [6, 8]. This is because away from the interfaces, $-L$ is a positive elliptic operator with a pseudodifferential square root. See [27] for a microlocal construction of this square root. Hence, the construction in [4, appendix A] applies, so for Cauchy data (f_0, f_1) (time $t=0$ say), the Cauchy to solution map may then be decomposed as

$$F(f_0, f_1) \equiv F^+ g_+ + F^- g_-, \quad \begin{bmatrix} g_+ \\ g_- \end{bmatrix} \equiv C \begin{bmatrix} f_0 \\ f_1 \end{bmatrix}$$

where C is a microlocally invertible matrix Ψ DO. The Cauchy data (g_+, g_-) may be interpreted as a single distribution g on a doubled space $\mathbf{Z} = Z_+ \sqcup Z_-$. The corresponding layers are then $\Omega_{\pm, j}$.

Combining the elastic parametrix construction in Rachele [6] with the scalar wave parametrix in the presence of singularities in the sound speed [4], we may construct a parametrix for R_T in regions where no glancing occurs at an interface. We will describe it as a sum of graph FIOs on \mathbf{Z} from sequences of reflections, transmissions, and P/S mode conversions, along with operators propagating data from one boundary to another, or propagating the initial data to boundary data.

4.1. Cauchy propagators

To begin, extend each restriction $\mu_j = \mu|_{\Omega_j}, \lambda_j = \lambda|_{\Omega_j}$ to a smooth function on \mathbb{R}^3 . Each $\eta \in T^* \Omega_{\pm, j}$ is associated with a unique P/S -bicharacteristic $\gamma_\eta^{P/S}(t)$ in $T^* \mathbb{R}^3$ passing through η at $t=0$, which may escape and possibly reenter $\Omega_{\pm, j}$, as $t \rightarrow \pm\infty$.

To prevent reentry of wavefronts, we introduce a pseudodifferential cutoff for P/S rays, $\phi^{P/S}(t, x, \xi)$, omitting some details for brevity. Let $t_{e\pm}^{P/S}, t_{r\pm}^{P/S}$ denote the first positive and negative escape and reentry times for the P/S -ray. We let $\phi^{P/S}(t, \gamma_\eta^{P/S}(t))$ be identically one on $[t_{e-}^{P/S}, t_{e+}^{P/S}]$ and supported in $(t_{r-}^{P/S}, t_{r+}^{P/S})$. One then modifies $\phi^{P/S}$ on a small neighborhood of $\mathbb{R} \times T^* \partial \Omega_{\pm, j}$ (the glancing P/S rays) to ensure it is smooth.

We then recall the construction of the Cauchy propagators E_j^\pm (with \pm corresponding to “forward” and “backward” propagators) described in detail in [6]: These are global FIOs that may be defined on each layer by smoothly extending the elastic parameters in the layer layer to all of \mathbb{R}^3 as done in [4]. These solution operators solve

$$\left. \begin{aligned} QE_j &\equiv 0 && \text{on } \mathbb{R}^3 \times (0, T) \text{ for } j = 0, 1 \\ E_0|_{t=0} &\equiv I, && E_1|_{t=0} \equiv 0 \\ \partial_t E_0|_{t=0} &\equiv 0, && \partial_t E_1|_{t=0} \equiv I \end{aligned} \right\} \text{mod smoothing.}$$

As Fourier integral operators, they are locally given by [6]

$$(E_j^\pm h)_l = \sum_{P/S} \sum_{m=1}^3 (2\pi)^{-3} \left[\int e^{i\phi_{P/S}^\pm} e_{P/S, \pm}^{lm} (j) \widehat{h}_{m, \pm}(\xi) d\xi \right] \tag{4.1}$$

with phase functions $\phi_{P/S}(t, x, \xi)$ and matrix-valued amplitudes $e_{P/S, \pm}^{lm}(t, x, \xi)$.

Finally, let J_{C-S} be the restriction of $\phi \circ E_j^\pm$ defined by

$$\phi \circ E_j^\pm := \sum_{P/S} \sum_{m=1}^3 (2\pi)^{-3} \phi^{P/S}(t, x, D_x) \left[\int e^{i\phi_{P/S}^\pm} e_{P/S, \pm}^{lm} \widehat{h}_{m, \pm}(\xi) d\xi \right]$$

to $\mathbb{R} \times \Omega_{\pm, j}$; this is the desired reflectionless propagator.

We also require a variant, denoted J_{C-S+} , of J_{C-S} in which waves travel only forward in time. For this, replace $\phi^{P/S}$ with some $\phi_+^{P/S}$ supported in $(t_{e-}^{P/S}, t_{e+}^{P/S})$ and equal to 1 on $[0, t_{e+}]$. Restricting J_{C-S+} to the boundary, we obtain the *Cauchy-to-boundary* map $J_{C-\partial} = J_{C-S+}|_{\mathbb{R} \times \partial Z}$. One may also construct the boundary-to-solution map, denoted $J_{\partial-S}$, analogous to the above using the construction in [8] for the smooth Lamé parameter case.

As in [4, Appendix], $J_{\partial-S}, J_{C-S+} \in I^{-1/4}(\mathbf{Z} \rightarrow \mathbb{R} \times \mathbf{Z})$, and $J_{C-\partial} \in I^0(\mathbf{Z} \rightarrow \mathbb{R} \times \partial \mathbf{Z})$. Also, $J_{\partial-S}, J_{C-S+}$ are parametrices for the elastic equation when applied to u such that $\text{WF}(u)$ lies in an open set $\mathcal{V} \subset T^*\mathbf{Z}$ whose P/S -bicharacteristics are disjoint from their respective glancing sets. Such a set exists since we are only concerned with a compact time interval and the glancing set is closed. The near-glancing covector set, denoted \mathcal{W} , is $T^*\mathbf{Z} \setminus \mathcal{V}$.

4.2. P/S-Mode projectors

Since we are in the elastic setting, it will be useful to define microlocal projections $\Pi_{P/S}$ that microlocally project an elastic wavefield u to the respective P and S characteristic sets. Locally and for small times, from (4.1), u has a representation

$$u = \sum_{P/S} \sum_{m=1}^3 (2\pi)^{-3} \left[\int e^{i\phi_{P/S}} e_{P/S}^{lm} \widehat{h}_m(\xi) d\xi \right]$$

and so we define

$$\Pi_P u = \sum_{m=1}^3 (2\pi)^{-3} \int e^{i\phi_P} e_P^{lm} \widehat{h}_m(\xi) d\xi.$$

Π_S is defined analogously. These definitions can be made global, although it is technically not necessary in our case since our analysis is done near the characteristic set of the elastic operator.

4.3. Boundary propagators

Outgoing solutions from boundary data $f \in \mathcal{D}'(\mathbb{R} \times \mathbf{Z})$ may be obtained by microlocally converting boundary data to Cauchy data, then applying $J_{C \rightarrow S}$ as explained in [4]. We give a cursory overview of the construction, which translates easily to the elastic setting. The boundary-to-Cauchy conversion can be achieved by applying a microlocal inverse of $J_{C \rightarrow \partial}$, conjugated by the time-reflecting map $S_s : t \rightarrow s - t$ for an appropriate s . Let $x = (x', x_3)$ be boundary normal coordinates near $\partial\Omega_{\pm,j}$. Near any covector $\beta = (t, x'; \tau, \xi') \in T^*\partial\Omega_{\pm,j}$ in the hyperbolic region $|\tau| > c_j^{P/S}|\xi'|$, there exists a unique P/S -bicharacteristic γ passing through β and lying inside $\Omega_{\pm,j}$ in some time interval $[s, t], s < t$.² Then $J_{\partial \rightarrow S}$ may be defined by $S_s J_{C \rightarrow S} J_{C \rightarrow \partial}^{-1} S_s$ microlocally near β . The inverse can be seen to exist microlocally away from glancing by “diagonalizing” the Cauchy propagators as done in [27] and applying the same construction of the scalar wave setting in [4].

On the elliptic region $|\tau| < c_j^{P/S}|\xi'|$ define $J_{\partial \rightarrow S}$ as a parametrix for the elliptic boundary problem. This may be constructed even in the systems setting as shown in [28]. Applying a microlocal partition of unity, we obtain a global definition of $J_{\partial \rightarrow S}$ away from a neighborhood of both P/S glancing regions $|\tau| = c_j^{P/S}|\xi'|$. It can be proven that $J_{\partial \rightarrow S} \in I^{-1/4}(\mathbb{R} \times \partial\mathbf{Z} \rightarrow \mathbb{R} \times \mathbf{Z})$. Its restriction to the boundary $r_{\partial} \circ J_{\partial \rightarrow S}$ consists of a pseudodifferential operator equal to the identity on \mathcal{W} and an elliptic graph FIO $J_{\partial \rightarrow \partial} \in I^0(\mathbb{R} \times \partial\mathbf{Z} \rightarrow \mathbb{R} \times \partial\mathbf{Z})$ describing waves traveling from one boundary to another.

4.4. Reflection and transmission

It is well known that transmitted and reflected waves arise from requiring a weak solution and its normal traction to be C^0 near the interface. Given incoming boundary data $f \in \mathcal{E}'(\mathbb{R} \times \partial\mathbf{Z}; \mathbb{C}^3)$ (an image of $J_{C \rightarrow \partial}$ or $J_{\partial \rightarrow \partial}$) microsupported near β , we seek data f_R, f_T satisfying the interface constraints

$$\begin{aligned} f + f_R &\equiv \iota f_T, \\ (\lambda_{\text{in}} \operatorname{div}(\nu J_{\partial \rightarrow S} \nu f + J_{\partial \rightarrow S} \nu f) + 2\mu_{\text{in}} \nabla_s(\nu J_{\partial \rightarrow S} \nu f + J_{\partial \rightarrow S} \nu f)) \cdot \eta|_{\mathbb{R} \times \partial\mathbf{Z}} \\ &\equiv \iota(\lambda_{\text{out}} \operatorname{div}(J_{\partial \rightarrow S} \nu f) + 2\mu_{\text{out}} \nabla_s J_{\partial \rightarrow S} \nu f) \cdot \eta|_{\mathbb{R} \times \partial\mathbf{Z}} \end{aligned}$$

Here, ν is time-reversal, so $\nu J_{\partial \rightarrow S} \nu$ is the outgoing solution that generated f . The map $\iota : \partial\mathbf{Z} \rightarrow \partial\mathbf{Z}$ reverses the copies of each boundary component within $\partial\mathbf{Z}$, and η denotes the unit normal vector to the interface in question. The subscripts *in* and *out* merely denote which side of the interface one is considering in the Lamé parameters.

The second equation above simplifies to a pseudodifferential equation

$$N_1 f + N_R f_R \equiv N_T f_T$$

with operators $N_1, N_R, N_T \in \Psi^1(\mathbb{R} \times \partial\mathbf{Z}; \mathbb{C}^3)$ that may be explicitly computed. The system may be microlocally inverted in the nonglancing, nonelliptic region³ to recover $f_R = M_R f, f_T = M_T f$ in terms of pseudodifferential reflection and transmission operators $M_R, \iota M_T \in \Psi^0(\mathbb{R} \times \partial\mathbf{Z}; \mathbb{C}^3)$. This is the analog to the reflection and transmission

²That is, its projection to $T^*\partial\Omega_{\pm,j}$ when it hits $\partial T^*\Omega_{\pm,j}$ is β , but we abuse notation.

³Here, the nonglancing, nonelliptic region refers to covectors in $T^*\Gamma$ that are not in the glancing set of either $\partial_t^2 - c_p^2 \Delta$ or $\partial_t^2 - c_s^2 \Delta$, but is in the hyperbolic set of one of the operators.

operators in the scalar wave case constructed in [13]. As in [13], a solution near an interface Γ_i (say) has the form $u = u_I + u_R + u_T$, with $u_I|_\Gamma = f, u_R|_\Gamma = f_R, u_T|_\Gamma = f_T$. The solution is such that u_I, u_R are microlocally zero on one side of the interface and u_T is zero on the other side. Computing M_R and M_T from the above interface conditions is rather messy, and it is difficult to see whether or not the above equations lead to an elliptic boundary problem for M_R, M_T since it involves a 6×6 matrix. Instead, in Appendix A we use the traction formulation of the interface conditions to compute these operators, which will allow us to use symplectic properties in order to compute and study the ellipticity of M_R and M_T .

Specifically, we prove in Appendix A

Lemma 4.1. *The operators M_R and M_T are well defined operators in $\Psi^0(\mathbb{R} \times \partial\mathbf{Z}; \mathbb{C}^3)$, microlocally elliptic in the jointly nonglancing, nonelliptic set on both sides of an interface.*

Significance and list of operators

Since we have so many symbols and operators, let us summarize them for quick reference.

Operator	Name	Summary
$J_{C \rightarrow S}$	Cauchy to solution operator	Propagator mapping Cauchy data to the corresponding solution of the homogeneous elastic wave equation.
$J_{C \rightarrow S+}$	forward Cauchy to solution operator	Similar to $J_{C \rightarrow S}$, but only propagates waves forward in time.
$J_{C \rightarrow \partial}$	Cauchy to boundary map	Restriction of $J_{C \rightarrow S}$ to the boundary, which includes each side of an interface.
$J_{C \rightarrow \partial+}$	forward Cauchy to boundary map	As $J_{C \rightarrow \partial}$, but with only waves that travel forward in time.
$J_{\partial \rightarrow S}$	boundary to solution map	Maps boundary data (associated with specific side of an interface) to a wave solution in the interior, traveling forward in time.
$\Pi_{P/S}$	P/S projectors	Microlocal projectors of an elastic wavefield u onto the P/S -characteristic set.
$J_{\partial \rightarrow \partial}$	boundary to boundary map	Restriction of $J_{\partial \rightarrow S}$ to the boundary (which includes interfaces). Hence, it propagates boundary data to the next boundary that the waves intersect.
$M_{R/T}$	reflection and transmission operators	Zeroth-order PsiDO's at the boundary that act as the reflection/transmission coefficients of the scattered wave from an incident field at an interface.

The construction of the parametrix is now taken directly from [4, Appendix].

4.5. Parametrix

First it will be convenient to define $M = M_R + iM_T$. With all the necessary components defined, we now set

$$\tilde{F} = J_{C \rightarrow S} + J_{\partial \rightarrow S} \sum_{k=0}^{\infty} (J_{\partial \rightarrow \partial} M)^k J_{C \rightarrow \partial} \tag{4.2}$$

$$\tilde{R}_{2T} = r_{2T} \circ \tilde{F}, \tag{4.3}$$

where r_{2T} is restriction to $t = 2T$. Again omitting the proof, it can be shown that $\tilde{F} \equiv F$ and $\tilde{R}_{2T} \equiv R_{2T}$ away from glancing rays. In the elastic case it means away from both

P and S glancing rays; that is, for initial data h_0 such that every broken bicharacteristic originating in $\text{WF}(h_0)$ is disjoint from the glancing set. Recalling that $M = M_R + M_T$, we may write \tilde{R}_{2T} as a sum of graph FIO indexed by sequences of reflections and transmissions:

$$\tilde{R}_{2T} = \sum_{s \in \{R, T\}^k, \lambda \in \{P, S\}^{k+1}} \tilde{R}_{s, \lambda}, \quad \tilde{R}_{()} = r_{2T} J_{C \rightarrow S} \tag{4.4}$$

$$\tilde{R}_{(s_1, \dots, s_k; \lambda_0, \dots, \lambda_k)} = r_{2T} J_{\partial \rightarrow S} \Pi_{\lambda_k} M_{s_k} J_{\partial \rightarrow \partial} \cdots \Pi_{\lambda_2} M_{s_2} J_{\partial \rightarrow \partial} \Pi_{\lambda_1} M_{s_1} J_{C \rightarrow \partial} \Pi_{\lambda_0}. \tag{4.5}$$

The solution operator \tilde{F} likewise decomposes into analogous components \tilde{F}_a .

Now that we have a parametrix, we will use it in the next section to obtain certain “subsurface” travel time and lens relations.

5. Proof of Theorem 1.1

In this section, we will prove our main result on the uniqueness of elastic wave speeds under the extended foliation condition.

A key ingredient in the proof of uniqueness will be the following theorem proved by Stefanov, Uhlmann, and Vasy in [3].

Theorem 5.1. *Choose a fixed metric g_0 on Ω . Let $n = \dim(\Omega) \geq 3$; let $c, \tilde{c} > 0$ be smooth, and suppose $\partial\Omega$ is convex with respect to both $g = c^{-2}g_0$ and $\tilde{g} = \tilde{c}^{-2}g_0$ near a fixed $p \in \partial\Omega$. If $d_g(p_1, p_2) = d_{\tilde{g}}(p_1, p_2)$ for p_1, p_2 on $\partial\Omega$ near p , then $c = \tilde{c}$ in Ω near p .*

We write down a trivial corollary due to continuity of the distance function.

Corollary 5.2. *Consider the same setup as in the above theorem. If $d_g(p_1, p_2) = d_{\tilde{g}}(p_1, p_2)$ for a dense set of points p_1, p_2 on some neighborhood of P in $\partial\Omega$, then $c = \tilde{c}$ in Ω near P .*

We need this since due to the multiple scattering in our setting, we will only be able to recover boundary travel times on a dense set of points and not a full neighborhood.

Outline of the proof of Theorem 1.1

The proof of the main theorem is technical but the main argument is quite intuitive and geometric. Thus, we provide a summary of the proof that emphasizes the key ideas. The first goal is to do a local recovery of the wave speeds in the form of Corollary 5.11. Inductively, suppose that we have recovered the Lamé parameters above $\Sigma_\tau, \tau > 0$, that is, inside Ω_τ^c , and let $z \in \Sigma_\tau$. Say we want to use Theorem 5.1 to recover c_P near z (a similar argument works for c_S). Viewing Σ_τ as the boundary of the domain Ω_τ , we would need to recover the local boundary distance function $d_P|_{\Sigma_\tau \times \Sigma_\tau}$ near z to apply the above theorem. Let $x \in \Sigma_P$ near z where Σ_P is the P -characteristic set defined earlier, $(x, \xi) \in S_{\Sigma_\tau}^* \bar{\Omega}_\tau$ pointing downward, and $l_{P, \tau}(x, \xi)$ the corresponding boundary travel time that we would like to recover.

Let γ be a purely transmitted P -bicharacteristic, entering Ω at some time $t < 0$ and passing through (x, ξ) at time $t = 0$. For convenience, let us view γ as lying in $T^*(\mathbb{R}^3 \times \mathbb{R}_t)$. With appropriate Cauchy data h_0 supported outside Ω , we can generate a microlocal P -wave whose wavefront set is initially along γ . Let us denote this wave solution by

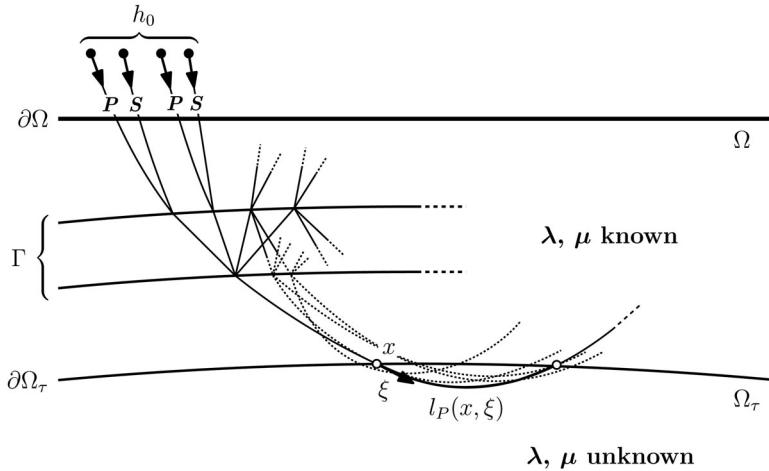


Figure 2. With knowledge of the elastic parameters above x , it is possible to construct initial data h_0 that produces a single P or S ray at almost every covector (x, ξ) , here a P ray. However, due to the presence of multiple reflected rays, it is not immediately possible to recover the length $l_{P,\tau}(x, \xi)$.

u_{h_0} . By propagation of singularities, u_{h_0} will have each point along γ in its wavefront set and two points in particular: $\gamma(0)$, which projects to $(x, \xi) \in T^*\mathbb{R}^3$, and $\gamma(l_{P,\tau}(x, \xi))$. The problem is that due to the interface and the multiple scattering of P/S -waves in the interior, their singularities will also lie in $\text{WF}(u_{h_0})$ so we cannot uniquely recover $\gamma(l_{P,\tau}(x, \xi))$ in this wavefront set. Hence, in addition to h_0 , we must microlocally construct additional Cauchy data (a “tail” similar to [4]) that eliminates this type of multiple scattering. In order to suppress mode converted transmissions resulting from the initial P -wave (see Figures 2 and 3), we will construct h_0 in two steps. In the first step, h_0 has wavefront set along a single covector associated to γ . We then modify h_0 with appropriate Cauchy data to eliminate mode converted transmissions when γ hits an interface. After this additional “tail” is constructed (see Proposition 5.3), we will be able to uniquely identify $\gamma(l_{P,\tau}(x, \xi))$ in the wavefront set.

In order to layer strip past an interface, we must also recover information on transmission angles for incident waves refracting from an interface (see Lemma 5.8). With the controls in place, we are able to first do a local recovery of the lens relations in the vicinity of a particular point and covector (Lemma 5.10) followed by a local recovery of the wave speeds near the point (Corollary 5.11). An inductive layer stripping argument leads to the global result whose proof is just below Corollary 5.11. Since Proposition 5.3 and Lemma 5.8 are tedious technical computations, the proofs of those are delegated to Appendices A and B.

We now turn to the multiple lemmas and propositions involved in proving the main theorem. Let $\mathcal{S} \subset T^*\Omega$ be the set of ξ such that every unused bicharacteristic belonging to a broken bicharacteristic through ξ is (+)-escapable (all definitions are in Appendix B). The set \mathcal{S} will be dense within an appropriate set, allowing us to work wholly inside \mathcal{S} (see Lemma 5.9). We will state a series of propositions and lemmas to prove the main theorem. We first state the following crucial proposition that is at the heart of proving our main theorem and whose proof requires the microlocal analysis of

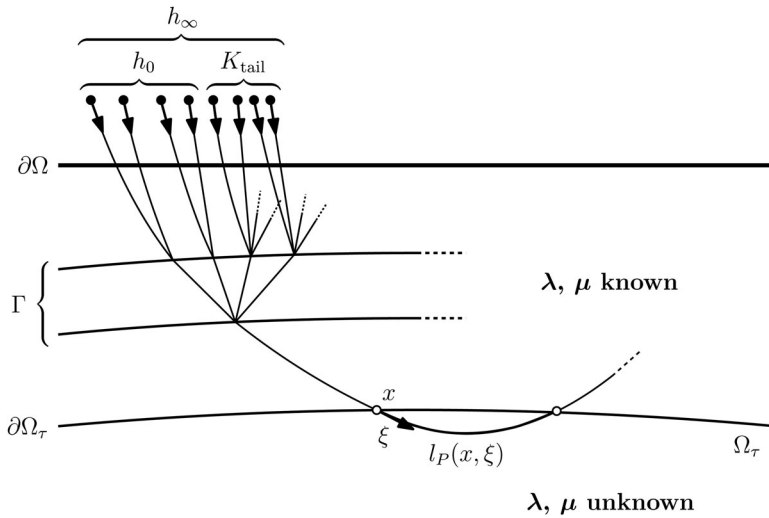


Figure 3. By appropriately augmenting the initial data h_0 with extra initial data K_{tail} , producing total initial data h_∞ , multiple reflections can be suppressed, and $l_{P,\tau}(x, \xi)$ can be recovered from examination of the solution’s wavefront set.

scattering control. In order to make the polarization statement in the following proposition, recall that $\Sigma_{P/S}$ are the characteristic varieties defined earlier.

Proposition 5.3. *Let $(x, \xi) \in \mathcal{S}, \tau = \rho(x)$, and let v be a distribution whose wavefront set is exactly $(x, \mathbb{R}_+\xi)$. Then there exists Cauchy data h_∞ supported outside Ω and a unique minimal time $T > 0$ such that $\text{WF}(R_T h_\infty - v) \cap T^*\Omega_\tau = \emptyset$ and $\text{WF}(R_{T+s} h_\infty - R_s v) \cap T^*\Omega_\tau = \emptyset$ for all $s \geq 0$. Moreover, we may choose v so that $\text{WF}(u_{h_\infty}) \subset \Sigma_P$ within Ω_τ for times t close enough to T . The same may be done with $\text{WF}(u_{h_\infty}) \subset \Sigma_S$ instead.*

Remark 5.4. The time T is very concrete. It is essentially a scalar multiple of the S-distance from x to $\partial\Theta$. The reason is that we need access to all S wave constituents starting near x that produce branches that eventually return to the surface $\partial\Theta$. The details will be made clear in the proof.

Remark 5.5. The second part of the theorem means that not only can we generate Cauchy data to produce a certain singularity at a given depth, but we may even construct it to be a P or an S wave. This is essential for the uniqueness result since we must be able to recover subsurface lens relations for the P and S speeds separately.

The proof of Proposition 5.3 is quite technical and is done in [Appendix B](#).

The next two lemmas are the main technical complications in the elastic setting. When we later show uniqueness via layer stripping, we will be able to layer strip past an interface if the wave speeds of both Lamé systems infinitesimally match up just past the interface, even when we do not have direct access past such an interface. More concretely, as we layer strip past an interface contained in Σ_τ (say), we will need to generate waves with wavefront at a fixed covector $\eta \in T^*_{\Sigma_\tau} \Omega_\tau$ (say) pointing inwards. To do this, we will need to generate an incoming wave that when it hits Σ_τ , it is singular at a covector related to η via Snell’s law of refraction so that the transmitted wave is singular

at η . In order to know this covector, which is needed to generate the appropriate incoming wave, we need to know the infinitesimal jumps in the wave speeds past an interface to obtain the correct transmission angles.

To do this, we rely on obtaining the principal symbols of reflection coefficients to recover the infinitesimal jumps in wave speeds past the interfaces. In terms of notation, any symbol with a tilde above it represents the corresponding symbol for the second set of Lamé parameters, and the superscript *prin* denotes the principal symbol of a pseudo-differential operator.

Lemma 5.6. *Suppose that $\Sigma_\tau \subset \Gamma$ and $c_{P/S} = \tilde{c}_{P/S}$ outside $\bar{\Omega}_\tau$. Assume $\mathcal{F} = \tilde{\mathcal{F}}$. Then*

$$M_R^{\text{prin}} = \tilde{M}_R^{\text{prin}} \text{ on } T^*\Sigma_\tau^-.$$

Remark 5.7. The proof actually shows that one may recover the full symbol, but it is unnecessary in our analysis.

Proof. This is essentially an inductive argument, whereby we recover the coefficients at each successive interface using appropriate sources. Let Σ_{τ_1} denote the first interface, and suppose $c_{P/S} = \tilde{c}_{P/S}$ outside Ω_{τ_1} . Since $\rho = 1$, both elastic operators Q, \tilde{Q} (see §2.1 for notation) agree on $\Omega_{\tau_1}^*$. Combining this with $\mathcal{F} = \tilde{\mathcal{F}}$, propagation of singularities, and the extended convex foliation assumption to ensure no trapped rays, then $u_h \equiv \tilde{u}_h$ in $\Omega_{\tau_1}^*$ for $h \in \mathbb{C}$.⁴ By taking a limit to $\Sigma_{\tau_1}^-$, we get $M_R J_{C \rightarrow \partial} h = \tilde{M}_R J_{C \rightarrow \partial} h$. By considering h to be a P wave and then picking h to be an S wave, we obtain the desired claim since we only need M_R in the hyperbolic regions where $J_{C \rightarrow \partial}$ is elliptic and so we may generate microlocal P and S waves at the first interface. The argument is a direct analogue to the one in [8, section 2.3].

To proceed inductively, suppose M_R is recovered for the first k interfaces $\Sigma_{\tau_1}, \dots, \Sigma_{\tau_{k-1}}$. Let Σ_{τ_k} be the k^{th} interface and let $(y, \eta) \in \partial^+ S^* \Omega_{\tau_k}$ be a fixed covector. We assume $c_{P/S} = \tilde{c}_{P/S}$ in $\Omega_{\tau_k}^*$ and so we may assume the transmission coefficients are recovered for these interfaces as well. We let $(x_0, \xi_0) \in T^* \Theta^*$ lie on the same P -transmitted ray as (y, η) which exists due to the convex foliation. We will repeat this construction for the S -transmitted ray too. Let h be Cauchy data supported in Θ^* whose wavefront set in $S^* \mathbb{R}^3$ is exactly (x_0, ξ_0) . The constituent of $\mathcal{F}h|_{\partial\Theta}$ associated to the first primary reflection from Σ_{τ_k} is

$$M_R (J_{\partial \rightarrow \partial} M_T)^{k-1} J_{\partial \rightarrow \partial} J_{C \rightarrow \partial} h.$$

Due to the extended convex foliation assumption, our assumptions on the wave speeds, and that $\mathcal{F}h = \tilde{\mathcal{F}}h$, we again have $u_h \equiv \tilde{u}_h$ on Ω_τ^* by propagation of singularities. Hence, the associated constituent for $\tilde{\mathcal{F}}h$ must be equal to this one at Σ_τ since we are not looking at what happens inside Ω_τ , as we are only considering a reflection. Since M_T^{prin} are the same for both operators on $\Sigma_{\tau_j}, j = 1, \dots, k - 1$, by our assumption on the wave speeds, then the same argument as before where we let h generate s waves associated to a purely transmitted s -ray through (y, η) shows $M_R^{\text{prin}}(y, \eta', \tau) =$

⁴In fact, we can use unique continuation to obtain the same result since we are allowed to measure outside Θ for an unlimited amount of time. Nevertheless, this is overkill for what we need here, which is a microlocal equivalence.

$\tilde{M}_R^{\text{prin}}(y, \eta', \tau)$ by applying the inverse of M_T^{prin} and of $J_{\partial \rightarrow \partial}^{\text{prin}}$. Here, (y, η') is the projection of (y, η) to $T^*\Sigma_{\tau_k}$. We are using the fact that, since the Lamé parameters match on $\Omega_{\tau_k}^*$, the operators $J_{\partial \rightarrow \partial}$ are equal as well for operators in this region. Also, these operators are elliptic near the hyperbolic point sets we are considering. \square

Lemma 5.8. *Suppose that $\Sigma_\tau \subset \Gamma, c_{P/S} = \tilde{c}_{P/S}$ outside $\bar{\Omega}_\tau$, and denote Σ_τ^\pm for the two sides of Σ_τ . Suppose that $M_R^{\text{prin}} = \tilde{M}_R^{\text{prin}}$ on $T^*(\mathbb{R}_t \times \Sigma_\tau^-)$. Then $c_{P/S} = \tilde{c}_{P/S}$ on Σ_τ^+ .*

The above lemma is essentially saying that the principal symbols of reflection coefficients are enough to recover the jumps in both wave speeds at an interface. This should not come as a surprise since the reflection coefficient would vanish identically if the speeds were actually continuous across the interface. Thus, two waves with identical reflections, must also have transmissions that correspond to the same covectors. Since the proof of the lemma is quite technical, we save it for the appendix (see Appendix A.1).

Both of these crucial lemmas will suffice to recover subsurface travel times and lens relations for a particular covector. We will show the following: Let $\Gamma \subset \Sigma_\tau$ be relatively open and let $T > 0$. Then the lens relations $(L_{P,\tau}, l_{P,\tau})$ and $(L_{S,\tau}, l_{S,\tau})$ are determined uniquely on the open sets of (x, ν) with $x \in \Gamma$ such that the unit speed geodesic issued from (x, ν) at time 0 in the metric $c_P^{-2}dx^2$, respectively $c_S^{-2}dx^2$, is transversal at x and hits Σ_τ again, transversely, at a point in Γ at a time not exceeding T and without hitting any other interfaces. Since we assume that the interfaces are not dense, one may always ensure with T or Γ small enough that such rays do not hit another interface before returning to Σ_τ .

Also, to recover the lens relation for a particular covector, we will need to use the microlocal scattering control in the form of Proposition 5.3. This requires covectors belonging to \mathcal{S} and we must ensure there are enough of them. The following lemma uses the extended convex foliation assumption to ensure that we have enough of them.

Lemma 5.9. *Let $x \in \Sigma_\tau$ for some τ . Then there is a neighborhood $B_x \subset \Sigma_\tau$ such that $B_x \cap \partial^+ S^* \Omega_\tau \cap \mathcal{S}$ is dense in $B_x \cap \partial^+ S^* \Omega_\tau$.*

Proof. The proof follows from the extended convex foliation condition and repeated application of Lemma B.8 and its proof.

Take a particular covector $(x, \nu) \in \partial^+ S^* \Omega_\tau$ pointing upwards and let $\gamma_{P/S, \nu_x}$ be the associated smooth bicharacteristic starting at (x, ν) . Considering γ_{P, ν_x} first, it will either glance or hit the next interface Σ_{τ_1} at time t_1 , say, transversely. If the latter, the convex foliation guarantees that both the P and S transmitted branches continuing γ_{P, ν_x} will also be transverse to Σ_{τ_1} and move “upward” (decreasing ρ). Also, there will be exactly two opposite branches at $\gamma_{P, \nu_x}(t_1)$ that are transverse to Σ_{τ_1} and move upward in backward time. If it glances, then by Lemma B.6, an arbitrary perturbation of ν avoids this. We can apply this analysis to each successive P branch discussed and iterate; since the time T_s in the definition of escapability is finite, there will be only finitely many branchings and so there will be a dense set of $\nu \in \partial^+ S_x^* \Omega_\tau$ such that all the P -branches of γ_{P, ν_x} escape. The continued S branches will be analyzed next.

Let us now consider γ_{S, ν_x} and use the same notation t_1 and Σ_{τ_1} as in the previous case. The analysis for γ_{S, ν_x} will apply just as well for the s branches discussed in the

previous paragraph. If $\gamma_{S, \nu_x}(t_1)$ does not glance, the extended convex foliation guarantees an S transmitted branch that continues γ_{P, ν_x} , is transverse to Σ_{τ_1} , and moves “upward” (decreasing ρ). The issue is that the transmitted P branch might be glancing where we have hit a critical angle. However, this glancing set is a dimension lower than the hyperbolic points and so we may perturb this P -branch to be transversal to Σ_{τ_1} and move upward. We may then continue this branch backward with an S ray that starts on $\partial^+ S^* \Omega_\tau$, is a slight perturbation of γ_{S, ν_x} , and has a different base point.

Hence, we now have both a transmitted P and S branch moving upwards by the extended convex foliation, and an opposite P and S branch moving upwards backward in time. We then apply the analysis in the last paragraph and iterate the above for each successive interface. Hence, either (x, ν) or an open set of perturbations of it will be escapable.

Using Lemma B.6, the above analysis shows there is a neighborhood $B_x \subset \Sigma_\tau$ of x such that a dense set of $\partial^+ S_{B_x}^* \Omega_\tau$ are escapable. Indeed, any covector that is not escapable can be perturbed by the above procedure. \square

In the following series of proofs, we rely on the previous lemma to keep using [Proposition 5.3](#) without explicitly saying so.

Lemma 5.10. *Let $(x, \xi) \in \partial T^* \Omega \cap S_+^* \Omega$ as described above and assume the extended convex foliation condition. If $\mathcal{F} = \tilde{\mathcal{F}}$ and $\lambda = \tilde{\lambda}, \mu = \tilde{\mu}$ outside Ω_τ , then $c_{P/S}$ and $\tilde{c}_{P/S}$ have identical subsurface lens relations w.r.t. Σ_τ in a neighborhood of (x, ξ) within $T_{\Sigma_\tau}^* \Omega$.*

Proof. Without loss of generality, under the extended convex foliation condition we may assume that x is a regular point since otherwise, one may use a continuity/density argument described in [5]. We will divide the proof into two cases, which have slightly different proofs.

The point x is not on an interface: We let $\nu \in \mathcal{E}'(\Omega)$ be such that $\text{WF}(\nu) = (x, \mathbb{R}\xi)$ and let h be as in [Proposition 5.3](#) supported outside Ω . We let $u = F(R_{-a}h)$ and $\tilde{u} = \tilde{F}(R_{-a}h)$ with an appropriately chosen a based on the support of h (see [5] for details). Now, $u = \tilde{u}$ outside Ω and by unique continuation, $u = \tilde{u}$ outside Ω_τ since the Lamé parameters coincide there. In fact all we need is that $u \equiv \tilde{u}$ in Ω_τ^* which follows by microlocal analysis. Indeed, any ray in this set has a branch that escapes Ω by the foliation condition. Thus, by propagation of singularities, $u = \tilde{u}$ inside Ω_τ^* modulo smoothing.

Let T denote the time the transmitted geodesic from $\partial\Omega$ reaches ξ . By [Proposition 5.3](#), we can ensure $\text{WF}(u)$ restricted to $\mathbb{R}_t \times \Omega_\tau$ is generated purely from the P -ray associated to ξ . We only consider those ξ whose associated P -geodesic does not encounter any interface before reaching $\partial\Omega_\tau$. This is always possible by the extended convex foliation condition and taking ξ that are near tangent to $\partial\Omega_\tau$. Since the Lamé parameters are smooth near x , then for a ξ nearly tangential to $\partial\Omega_\tau$, the first singularity of u in $T_+^* \Sigma_\tau$ occurs at time $T + l_{P, \tau}(x, \xi)$ and covector $L_{P, \tau}(x, \xi)$. This must be true for \tilde{u} as well since $u = \tilde{u}$ outside Ω_τ . Hence, $l_{P, \tau}(x, \xi) = \tilde{l}_{P, \tau}(x, \xi)$ and $L_{P, \tau}(x, \xi) = \tilde{L}_{P, \tau}(x, \xi)$. We then repeat the above argument using [Proposition 5.3](#) to generate a pure s -wave, singular precisely at (x, ξ) when restricted to Ω_τ at the appropriate time. This works since we can always restrict to rays which do not hit any interfaces before returning to Σ_τ by the convex foliation.

The point x is at an interface: First, without loss of generality, we may assume that Σ_τ actually coincides with the interface near x . Indeed, any point $z \in \Sigma_\tau$ near x that is not at an interface implies that the Lamé parameters are smooth there. Hence we may apply the above result for the smooth case combined with [Theorem 5.1](#) to show that the wave speeds coincide near such points. Progressing in this fashion shows that both wave speeds in fact coincide near x up to the interface that contains x , and so the wavefields coincide there as well. Hence, we may assume that Σ_τ is the interface.

Using [Lemma 5.8](#), we conclude that if u is a pure P -wave for some time in Ω_τ , then \tilde{u} is as well, both associated to (x, ξ) , even though inside Ω_τ they could theoretically be quite different.

We then examine the construction of h in [Proposition 5.3](#) more closely.⁵ The P/S -directly transmitted component of $R_T h$ is $\mathbf{DT}_{k,P/S}^+ h$ from definition B.1. We make the decomposition $h = h_0 + K_{\text{tail}}$. We take any wavefield v , supported in Ω_τ initially and whose wavefront set is exactly $\gamma_{\xi_x}^P$ and inside Σ_P , where $\gamma_{\xi_x}^P$ is a P -bicharacteristic whose initial covector is $(x, \xi) \in T^* \Omega_\tau$. With ρ_Γ denoting restriction to Γ , we may view $\rho_{\Sigma_\tau} v$ as boundary data. The construction of h_0 and K_{tail} in [Proposition 5.3](#) ensures

$$R_{T+t} h|_{\Omega_\tau} \equiv \mathbf{DT}_{k,P}^+ h_0|_{\Omega_\tau} \equiv v|_{\Omega_\tau}.$$

That is, the directly transmitted constituent of h inside Ω_τ (the “underside” of Σ_τ) is precisely a P -wave associated to (x, ξ) . The point is that the same initial data h will also produce a pure P -wave with respect to \tilde{c}_P on the underside of Σ_τ by [Lemma 5.8](#) since that lemma implies that $\tilde{M}_T \equiv M_T$ at $T^* \Sigma_\tau$ near x .

Thus, since the transmission matrices of u and \tilde{u} coincide microlocally near (x, ξ) , then

$$\rho_{\Sigma_\tau} v \equiv \rho_{\Sigma_\tau} u|_{\Omega_\tau} \equiv \mathbf{DT}_{k,P}^+ h = \widetilde{\mathbf{DT}}_{k,P}^+ h \equiv \rho_{\Sigma_\tau} \tilde{u}|_{\Omega_\tau}.$$

We note that inside Ω_τ , $\tilde{J}_{\partial \rightarrow S} \rho_{\Sigma_\tau} v$ is indeed a pure P -wave associated to (x, ξ) , so $\tilde{J}_{\partial \rightarrow S} \rho_{\Sigma_\tau} \tilde{u}$ will be as well with speed \tilde{c}_P . By our assumptions, $\text{WF}(u|_{\Sigma_\tau^-}) = \text{WF}(\tilde{u}|_{\Sigma_\tau^-})$. By [Proposition 5.3](#), if we consider the t -component of this wavefront set, then the first t past T in this wavefront set will be precisely $l_{P,\tau}(x, \xi)$ by our construction. By equality of the wavefields and since \tilde{u} was also a pure P -wave at time T in Ω_τ , then $l_{P,\tau}(x, \xi) = \tilde{l}_{P,\tau}(x, \xi)$. A similar argument lets us conclude $l_{S,\tau}(x, \xi) = \tilde{l}_{S,\tau}(x, \xi)$ as well. \square

We can combine the above lemma with [Theorem 5.1](#) to obtain the key corollary. First, let $d_{P/S}^\tau$ denote the P/S -distance function restricted to $\bar{\Omega}_\tau \times \bar{\Omega}_\tau$.

Corollary 5.11. *With the assumptions in the above lemma, $d_{P/S}^\tau|_{\Sigma_\tau \times \Sigma_\tau} = \tilde{d}_{P/S}^\tau|_{\Sigma_\tau \times \Sigma_\tau}$ in some neighborhood of x , and $c_{P/S} = \tilde{c}_{P/S}$ in some neighborhood of x .*

Proof of Theorem 1.1. The proof is by contradiction. Suppose $c_P \neq \tilde{c}_P$ or $c_S \neq \tilde{c}_S$, and let $f = |c_P - \tilde{c}_P|^2 + |c_S - \tilde{c}_S|^2$. Now consider $S := \Omega_\tau \cap \text{supp } f$, and take $\tau = \min_S \rho : \text{so}$

⁵The following argument is necessary to ensure that we match a P travel time associated to Q with the corresponding one associated to \tilde{Q} rather than an s travel time associated to \tilde{Q} .

$c_P = \tilde{c}_P$ and $c_S = \tilde{c}_S$ above Ω_τ , but by compactness there is a point $x \in \Sigma_\tau \cap S$. The condition that $\rho^{-1}(T)$ has measure zero rules out the trivial case $\tau = \tau_0$.

Let us now consider a small neighborhood of x , denoted B_x , and we consider the Σ_τ -boundary distance function $d_{P/S}^\tau$ restricted to such neighborhoods. Since the interfaces are not dense, and we assume convex foliation, we may choose B_x small enough so that all P and S rays corresponding to rays staying completely inside B_x do not reach an interface; i.e. even the mode converted rays do not reach an interface. This insures that a P -wave that hits B_x , transmits a P and S wave, the P -wave returns to Σ_τ first before any other ray.

We now consider two cases, depending on whether x is on an interface of $c_{P/S}$ or not.

Smooth case: $x \notin \Gamma$. As in [3] we use the fact that $c_{P/S}$ and $\tilde{c}_{P/S}$ are equal above Ω_τ to show they locally have the same lens relation on Σ_τ . We can then apply [Corollary 5.11](#) to show that in fact $c_{P/S} = \tilde{c}_{P/S}$ near x , contradicting $x \in \text{supp} f$. The additional wrinkle is that we must ensure that $\tilde{c}_{P/S}$ is also smooth near x .

Suppose on the contrary that $\tilde{c}_{P/S}$ were not smooth at x . Since $c_{P/S} = \tilde{c}_{P/S}$ on Ω_τ^* , if $x \in \bar{\Gamma}$, then $\bar{\Gamma}$ must be tangent to the leaf Σ_τ . Now let γ be any bicharacteristic through a covector (x, ξ) not tangential to the leaf Σ_τ , and choose initial data h by [Proposition 5.3](#) satisfying $\text{WF}(R_T h) = (x, \mathbb{R}_+ \xi)$. Then $\tilde{u}_h(T)$ is singular on the reflected bicharacteristic to γ at x . This is because the reflection operator M_R is elliptic at $T^* \bar{\Gamma}$ in the non-glancing region as shown in [Appendix A](#). But this is impossible, since $u_h(T) = \tilde{u}_h(T)$ on $\Omega_{\tau_2}^*$ and the reflected bicharacteristic is contained in Ω_τ^* for t slightly greater than T , since $\bar{\Gamma}$ is tangent to Σ_τ .

From the argument above, we conclude $c_{P/S}, \tilde{c}_{P/S}$ are smooth in a sufficiently small ϵ -ball $B_\epsilon(x)$. Next, there exists a smaller neighborhood $B_{\epsilon'}(x) \subset B_\epsilon(x)$ in which every two points have a minimal-length path between them that is contained in $B_\epsilon(x)$, and in particular does not intersect $\Gamma \cup \bar{\Gamma}$. This is true by the boundedness of $c_{P/S}$ and $\tilde{c}_{P/S}$. Namely, picking global bounds $0 < m < c_{P/S}, \tilde{c}_{P/S} < M$ and taking $\epsilon' = \epsilon m / (m + 2M + 1)$, one can verify $d_{P/S}(y, \partial B_{\epsilon'}(x)) > 2 \text{diam}_P B_{\epsilon'}(x)$.

Finally, we apply [Lemma 5.10](#), concluding that $c_{P/S}$ and $\tilde{c}_{P/S}$ have identical lens relations for covectors $(x, \xi) \in \partial T^* \Omega_\tau \cap T^* \Omega$ whose bicharacteristics do not intersect any interfaces before returning to Σ_τ . Note that the lemma is applied multiple times to recover the lens relation for each wave speed. This is true, in particular, for the geodesics connecting points in $U = B_{\epsilon'}(x) \cap \Sigma_\tau$. Hence, $d_{P/S}^\tau = \tilde{d}_{P/S}^\tau$ on $U \times U$. Applying local boundary rigidity ([Theorem 5.1](#) and its corollary), we conclude $c_P = \tilde{c}_P$ and $c_S = \tilde{c}_S$ on some neighborhood of x , contradicting $x \in \text{ess supp } f$.

Interface case: $x \in \Gamma$. This follows from the above case using [Lemma 5.8](#) and [Lemma 5.10](#). Indeed, it is those two lemmas that allow us to recover the lens relation on the underside of the interface Σ_τ for both c_P and c_S . Similarly to the smooth case, the ball $B_\epsilon(x)$ is constructed to be disjoint from any other interface except for Σ_τ so that rays between points in Σ_τ , starting on the underside of Σ_τ in $B_\epsilon(x)$ stay completely in $B_\epsilon(x)$ before returning to Σ_τ . □

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A. Computation of reflection and transmission PsiDO's

Let us recall the traction formulation of the elastic equation, and since we work locally, we assume for simplicity at first that an interface Γ_i is given by $\{x_3 = 0\}$. The non-flat case will not require much more work, and we provide details on this later. A similar setup with analogous calculations may be found in [29].

Set the unit normal to interface Γ_i by $\nu = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Then the *traction components* are defined to be

$$\mathbf{t}_j = (\lambda \operatorname{div} \otimes \operatorname{Id} + 2\mu \widehat{\nabla})u \cdot e_j, \quad j = 1, 2, 3$$

where e_1, e_2, e_3 are the standard basis vectors in \mathbb{R}^3 , and the elastic equation reads

$$\partial_i^2 u = \operatorname{div} \begin{bmatrix} \mathbf{t}_1 \\ \mathbf{t}_2 \\ \mathbf{t}_3 \end{bmatrix} = \partial_{x_1} \mathbf{t}_1 + \partial_{x_2} \mathbf{t}_2 + \partial_{x_3} \mathbf{t}_3 \quad (\text{A.1})$$

It is also useful to denote the tangential dual variables in space by $\zeta' = (\zeta_1, \zeta_2, 0)$. Locally, the PDE can be put into the form

$$\partial_{x_3} \begin{bmatrix} u \\ \mathbf{t}_3 \end{bmatrix} = A(t, x', D_t, D'_x) \begin{bmatrix} u \\ \mathbf{t}_3 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{11}^T \end{bmatrix} \begin{bmatrix} u \\ \mathbf{t}_3 \end{bmatrix} \tag{A.2}$$

Although this is natural, A will be a 6×6 matrix of PsiDO's of different order, which does not allow using the PsiDO calculus. Thus, it is convenient to instead write the partial differential equation using $\begin{bmatrix} \langle D_t \rangle u \\ \mathbf{t}_3 \end{bmatrix}$ with $\langle D_t \rangle := (1 + D_t^2)^{1/2}$; using the same symbols, we have

$$\partial_{x_3} \begin{bmatrix} \langle D_t \rangle u \\ \mathbf{t}_3 \end{bmatrix} = A(t, x', D_t, D'_x) \begin{bmatrix} \langle D_t \rangle u \\ \mathbf{t}_3 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{11}^T \end{bmatrix} \begin{bmatrix} \langle D_t \rangle u \\ \mathbf{t}_3 \end{bmatrix} \tag{A.3}$$

Here, the pseudodifferential operators A_{ij} are all in $\Psi^1(\mathbb{R} \times \Gamma)$ and have principal symbols

$$\begin{aligned} a_{11} &= \frac{1}{i} \begin{bmatrix} 0 & 0 & \xi_1 \\ 0 & 0 & \xi_2 \\ \alpha \xi_1 & \alpha \xi_2 & 0 \end{bmatrix} & a_{12} &= \tau \begin{bmatrix} \mu^{-1} & 0 & 0 \\ 0 & \mu^{-1} & 0 \\ 0 & 0 & (\lambda + 2\mu)^{-1} \end{bmatrix} \\ a_{21} &= \frac{1}{\tau} \begin{bmatrix} \beta_1 \xi_1^2 + \mu \xi_2^2 - \tau^2 & \xi_1 \xi_2 \beta_2 & 0 \\ \xi_1 \xi_2 \beta_2 & \mu \xi_1^2 + \beta_1 \xi_2^2 - \tau^2 & 0 \\ 0 & 0 & -\tau^2 \end{bmatrix} & a_{22} &= a_{11}^T \end{aligned}$$

where

$$\alpha = \frac{\lambda}{\lambda + 2\mu} \quad \beta_1 = 4\mu \frac{\lambda + \mu}{\lambda + 2\mu} \quad \beta_2 = \mu \frac{3\lambda + 2\mu}{\lambda + 2\mu}.$$

Also, the eigenvectors of the principal symbol of A are easy to find. Indeed, we can first write

$$\mathbf{t}_j = \mathcal{T}_j(x, D_x)u$$

where \mathcal{T}_j is a Ψ DO of order 1 with principal symbol denoted $\tilde{\mathbf{t}}_j(x, \xi)$.

Explicitly, one may compute

$$\tilde{\mathbf{t}}_3(x, \xi) = i \begin{bmatrix} \mu \xi_3 & 0 & \mu \xi_1 \\ 0 & \mu \xi_3 & \mu \xi_2 \\ \lambda \xi_1 & \lambda \xi_2 & (\lambda + 2\mu) \xi_3 \end{bmatrix}.$$

To do principal symbol computations (this is enough since obtaining the lower order terms is quite standard in the literature) we employ the correspondence between $\partial_t, \partial_{x_1}, \partial_{x_2}$ and $i\tau, i\xi_1, i\xi_2$, respectively.

If $u^{(j)}$, $j = 1, 2$ represents u on each side of the interface, then the interface conditions become simply

$$\begin{aligned} u^{(1)} &= u^{(2)} && \text{on } \Gamma \\ \mathbf{t}_3^{(1)} &= \mathbf{t}_3^{(2)} && \text{on } \Gamma. \end{aligned}$$

Let us denote $U = [u \ \mathbf{t}_3]^\top$, so the interface condition is briefly stated as

$$U^{(1)} = U^{(2)} \text{ on } \Gamma.$$

As shown in [27, section 3] and [7], there is an elliptic matrix pseudodifferential operator in Ψ^0 denoted $S(x, D', D_t)$ with microlocal inverse S^- that diagonalizes A :

$$A(x, D', D_t) = S(x, D', D_t) \text{diag}(C_p^I, C_s^I, C_p^O, C_s^O) S^-(x, D', D_t)$$

where $C_p^{I/O}$ are scalar operators in Ψ^1 corresponding to the incoming (I) and outgoing (O) P waves, and $C_S^{I/O}$ is a diagonal 2×2 pseudodifferential operator corresponding to the incoming and outgoing S waves.

We can denote the columns of S by $S_{p/S, \pm}$ in correspondence with the diagonal matrix, so that the modes are exactly the six components of the vector valued distribution

$$V := S^{-1}U$$

where S^{-1} is a parametrix for S . Denote by $S^{(1)}, S^{(2)}$ the matrix S on either side of the interface.

With this notation, the interface conditions read

$$S^{(1)}V^{(1)} = S^{(2)}V^{(2)} \text{ on } \Gamma,$$

where it is important to note that because x_3 changes sign across Γ , the first three components of $V^{(1)}$ are incoming while those of $V^{(2)}$ are outgoing. We proceed to compute the principal symbol of S and its microlocal inverse.

Let v be an eigenvector of the principal symbol p of the elastic wave operator, with corresponding eigenvalue $\tau^2 - c_S^2|\xi|^2$ or $\tau^2 - c_p^2|\xi|^2$ so that using (A.1), one has

$$\begin{aligned} pv &= (\tau^2 - c_{p/S}^2|\xi|^2)v \iff (\tau^2 \text{Id} + i\xi_1 \tilde{\mathbf{t}}_1 + i\xi_2 \tilde{\mathbf{t}}_2 + i\xi_3 \tilde{\mathbf{t}}_3)v = (\tau^2 - c_{p/S}^2|\xi|^2)v \\ &\iff (\tau^2 \text{Id} + i\xi_1 \tilde{\mathbf{t}}_1 + i\xi_2 \tilde{\mathbf{t}}_2 \pm i\xi_3 \tilde{\mathbf{t}}_3)v = 0 \text{ if we set } \xi_3 = \pm \xi_{3, p/S} := \\ &\pm \sqrt{c_{p/S}^2 \tau^2 - |\xi'|^2} \pm i\xi_{3, p/S} \tilde{\mathbf{t}}_3 v = -(\tau^2 \text{Id} + i\xi_1 \tilde{\mathbf{t}}_1 + i\xi_2 \tilde{\mathbf{t}}_2)v = a_{21}\tau v + a_{22}\tilde{\mathbf{t}}_3(v). \end{aligned}$$

Thus, $\begin{bmatrix} \tau v \\ \tilde{\mathbf{t}}_3(v) \end{bmatrix}$ is an eigenvector of the principal symbol of A with eigenvalues $\pm i\xi_{3, p/S}$ when v is in the null space of p , which is where propagation occurs. It is well known that there are six independent eigenvectors with four eigenvalues [29].

It will be useful to denote these normal momenta components by

$$a_{p/S} = \sqrt{c_{p/S}^2 \tau^2 - |\xi'|^2}$$

where we later use a superscript to distinguish which side of the interface we are considering. During these computations, we assume that $a_{p/S}$ is a positive real number; that is, we are in the joint hyperbolic regime for both wave speeds. One can analyze the elliptic regions as well with a sign change in the following computations. Hence, we may form the 6×6 matrix of eigenvectors

$$S = \begin{bmatrix} | & | & | & | & | & | \\ s_{p,I} & s_{sV,I} & s_{sH,I} & s_{p,O} & s_{sV,O} & s_{sH,O} \\ | & | & | & | & | & | \end{bmatrix}$$

that we compute explicitly below and corresponding eigenvalue diagonal matrix

$$\Lambda = \text{diag}(i\xi_{3,p}, i\xi_{3,S}, i\xi_{3,S}, -i\xi_{3,p}, -i\xi_{3,S}, -i\xi_{3,S}).$$

By construction, one has

$$AS = S\Lambda.$$

Then form

$$K = \begin{bmatrix} 0_3 & I_3 \\ I_3 & 0_3 \end{bmatrix}.$$

We will compute S^{-1} and see that it has a particularly simple form despite S being a large, complicated matrix. Label V_{\pm} as the 3×3 matrix of eigenvectors of p , which in particular are in the null space of $p(t, x)$ on the respective characteristic set. More explicitly, one has

$$V_{\pm} = \begin{bmatrix} \xi_1 & \pm \xi_1 a_S & -\xi_2 \\ \xi_2 & \pm \xi_2 a_S & \xi_1 \\ \mp a_P & |\xi'|^2 & 0 \end{bmatrix}.$$

With the eigenvectors of S computed above, it is convenient to define

$$\tilde{t}_3 V_{\pm} = i \begin{bmatrix} \mp 2\mu \xi_1 a_P & -2\mu \xi_1 \chi & \pm \mu a_S \xi_2 \\ \mp 2\mu \xi_2 a_P & -2\mu \xi_2 \chi & \mp \mu a_S \xi_1 \\ 2\mu \chi & \mp 2\mu |\xi'|^2 a_S & 0 \end{bmatrix}$$

where $\chi = \frac{1}{2} \tau^2 / \mu - |\xi'|^2$.

Thus, with our earlier computation of the eigenvectors (A.4), we have

$$S = \begin{bmatrix} \tau V_+ & \tau V_- \\ \tilde{t}_3 V_+ & \tilde{t}_3 V_- \end{bmatrix}.$$

Note, one can easily scale the components of S to make S order 0 by dividing by powers of τ , which is nonzero near the characteristic set. We do not do this in order to have a cleaner computation.

Hence, we have

$$S = \begin{bmatrix} \tau \xi_1 & \tau \xi_1 a_S & -\xi_2 & \tau \xi_1 & -\tau \xi_1 a_S & -\tau \xi_2 \\ \tau \xi_2 & \tau \xi_2 a_S & \tau \xi_1 & \tau \xi_2 & -\tau \xi_2 a_S & \tau \xi_1 \\ -\tau a_P & \tau |\xi'|^2 & 0 & \tau a_P & \tau |\xi'|^2 & 0 \\ -2\mu \xi_1 a_P & -2\mu \xi_1 \chi & \mu a_S \xi_2 & 2\mu \xi_1 a_P & -2\mu \xi_1 \chi & -\mu a_S \xi_2 \\ -2\mu \xi_2 a_P & -2\mu \xi_2 \chi & -\mu a_S \xi_1 & 2\mu \xi_2 a_P & -2\mu \xi_2 \chi & \mu a_S \xi_1 \\ 2\mu \chi & -2\mu |\xi'|^2 a_S & 0 & 2\mu \chi & 2\mu |\xi'|^2 a_S & 0 \end{bmatrix}$$

And

$$S^T K = \begin{bmatrix} -2\mu \xi_1 a_P & -2\mu \xi_2 a_P & 2\mu \chi & \tau \xi_1 & \tau \xi_2 & -\tau a_P \\ -2\mu \xi_1 \chi & -2\mu \xi_2 \chi & -2\mu |\xi'|^2 a_S & \tau \xi_1 a_S & \tau \xi_2 a_S & \tau |\xi'|^2 \\ \mu a_S \xi_2 & -\mu a_S \xi_1 & 0 & -\tau \xi_2 & \tau \xi_1 & 0 \\ 2\mu \xi_1 a_P & 2\mu \xi_2 a_P & 2\mu \chi & \tau \xi_1 & \tau \xi_2 & \tau a_P \\ -2\mu \xi_1 \chi & -2\mu \xi_2 \chi & 2\mu |\xi'|^2 a_S & -\tau \xi_1 a_S & -\tau \xi_2 a_S & \tau |\xi'|^2 \\ -\mu a_S \xi_2 & \mu a_S \xi_1 & 0 & -\tau \xi_2 & \tau \xi_1 & 0 \end{bmatrix}$$

A quick calculation shows

$$S^T K S = \tau \text{diag}(-2\tau^2 a_P, -2\tau^2 |\xi'|^2 a_S, -2\tau^2 a_S, 2\tau^2 a_P, 2\tau^2 |\xi'|^2 a_S, 2\tau^2 a_S) := D.$$

Thus, $S^{-1} = D^{-1} S^T K$.

Next it is useful to define

$$\tilde{E} = \begin{bmatrix} \xi_1 & \xi_2 & 0 \\ 0 & 0 & 1 \\ -\xi_2 & \xi_1 & 0 \end{bmatrix}, \quad \tilde{E}^{-1} = \frac{1}{|\xi'|^2} \begin{bmatrix} \xi_1 & 0 & -\xi_2 \\ \xi_2 & 0 & \xi_1 \\ 0 & |\xi'|^2 & 0 \end{bmatrix}.$$

And set $E = \begin{bmatrix} \tilde{E} & 0 \\ 0 & \tilde{E} \end{bmatrix}$. Without loss of generality, we assume $\tau = 1$ since τ is a nonzero constant along bicharacteristics and can be used as a scaling factor. Then

$$ES = \begin{bmatrix} |\xi'|^2 & |\xi'|^2 a_S & 0 & |\xi'|^2 & -|\xi'|^2 a_S & 0 \\ -a_P & |\xi'|^2 & 0 & a_P & |\xi'|^2 & 0 \\ 0 & 0 & |\xi'|^2 & 0 & 0 & |\xi'|^2 \\ -2\mu |\xi'|^2 a_P & -2\mu |\xi'|^2 \chi & 0 & 2\mu |\xi'|^2 a_P & -2\mu |\xi'|^2 \chi & 0 \\ 2\mu \chi & -2\mu |\xi'|^2 a_S & 0 & 2\mu \chi & 2\mu |\xi'|^2 a_S & 0 \\ 0 & 0 & -\mu |\xi'|^2 a_S & 0 & 0 & \mu |\xi'|^2 a_S \end{bmatrix}$$

Then

$$(ES)^{-1} = S^{-1}E^{-1} = D^{-1}S^TKE^{-1}$$

and

$$S^TKE^{-1} = \begin{bmatrix} -2\mu a_p & 2\mu\chi & 0 & 1 & -a_p & 0 \\ -2\mu\chi & -2\mu|\xi'|^2 a_s & 0 & a_s & |\xi'|^2 & 0 \\ 0 & 0 & -\mu a_s & 0 & 0 & 1 \\ 2\mu a_p & 2\mu\chi & 0 & 1 & a_p & 0 \\ -2\mu\chi & 2\mu|\xi'|^2 a_s & 0 & -a_s & |\xi'|^2 & 0 \\ 0 & 0 & \mu a_s & 0 & 0 & 1 \end{bmatrix}.$$

Note that if we are away from normal incidence so that $|\xi'|^2 \neq 0$, we may use this as an elliptic scaling factor as well to make S and E order 0 as long as we make A order 1. Then, we may remove all instances of $|\xi'|$ appearing in the above formulas and replace the appearance of τ with $\hat{\tau} = \tau/|\xi'|$. Denoting the interface as Γ , we recall that $U^{(i)} = S^{(i)}V^{(i)}$ are microlocal solutions to the PDE on opposite side of the interface with interface conditions given by

$$S^{(1)}V^{(1)} = S^{(2)}V^{(2)} \text{ on } \Gamma$$

Now the first three components of $V^{(1)}$ represent incident “incoming” waves, denoted $v_I^{(1)}$, and the latter three components reflected “outgoing” waves using the ansatz

$$V^{(1)} = \begin{bmatrix} v_I^{(1)} \\ Rv_I^{(1)} \end{bmatrix}, \quad V^{(2)} = \begin{bmatrix} Tv_I^{(1)} \\ 0 \end{bmatrix}.$$

Thus, we obtain

$$\begin{bmatrix} v_I^{(1)} \\ Rv_I^{(1)} \end{bmatrix} = (S^{(1)})^{-1}S^{(2)} \begin{bmatrix} Tv_I^{(1)} \\ 0 \end{bmatrix} := Q \begin{bmatrix} Tv_I^{(1)} \\ 0 \end{bmatrix}.$$

So writing $Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix}$, where each entry is a 3×3 block matrix, we obtain the two equations

$$I = Q_{11}T \text{ and } R = Q_{21}T$$

So if we have Q_{11} being microlocally invertible, we would obtain $T = Q_{11}^{-1}$ and $R = Q_{21}Q_{11}^{-1}$. Notice that $(S^{(1)})^{-1}S^{(2)} = D^{-1}((S^{(1)})^TKE^{-1})(ES^{(2)})$ so it will suffice to show that $[(S^{(1)})^TKE^{-1})(ES^{(2)})]_{11}$ (the first 3×3 subblock) is invertible.

A.1. Defining $M_{R/T}$

In order to define $M_{R/T}$, note that if we view the 6×6 matrix S as four 3×3 blocks labeled $S_{ij}^{(1)}$, then $S_{11}^{(1)}$ and $S_{12}^{(1)}$ are invertible. On the “upper” side of the interface, we have incoming wave u_I with an outgoing reflected wave u_R . When restricted to Γ , we have $u_I = S_{11}^{(1)}V_I$. Since $U^{(1)} = S^{(1)}V^{(1)}$, then the solution is $u^{(1)} = S_{11}^{(1)}v_I + S_{12}^{(1)}Rv_I = u_I + u_R$ with u_I and u_R defined respectively by that equation. Thus, we have $u_R = S_{12}^{(1)}R(S_{11}^{(1)})^{-1}u_I|_\Gamma$ and the natural definition is $M_R = S_{12}^{(1)}R(S_{11}^{(1)})^{-1}$ which is elliptic if R is elliptic. Similarly, $u^{(2)} = S_{11}^{(2)}Tv_I = S_{11}^{(2)}T(S_{11}^{(1)})^{-1}u_I$ so that $M_T := S_{11}^{(2)}T(S_{11}^{(1)})^{-1}$.

Proof of Lemma 4.1. By the above construction, it suffices to show R and T are elliptic. By looking at the structure of ES and S^TKE^{-1} , namely that each of the four subblocks have a block structure consisting of a 2×2 matrix, and a 1×1 matrix, and the 1×1 pieces are trivial, it will

suffice to analyze the remaining 2×2 constituents. Then the first 2×2 minor of this matrix is given by the multiplication of

$$\begin{bmatrix} -2\mu_1 a_p^{(1)} & 2\mu_1 \chi^{(1)} & 1 & -a_p^{(1)} \\ -2\mu_1 \chi^{(1)} & -2\mu_1 a_s^{(1)} & a_s^{(1)} & 1 \end{bmatrix} \begin{bmatrix} 1 & a_s^{(2)} \\ -a_p^{(2)} & 1 \\ -2\mu_2 a_p^{(2)} & -2\mu_2 \chi^{(2)} \\ 2\mu_2 \chi^{(2)} & -2\mu_2 a_s^{(2)} \end{bmatrix} = \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix}.$$

So

$$\begin{aligned} t_{11} &= -2\mu_1 a_p^{(1)} - 2\mu_1 \chi^{(1)} a_p^{(2)} - 2\mu_2 a_p^{(2)} - 2\mu_2 \chi^{(2)} a_p^{(1)} \\ &= -2\mu_1 a_p^{(1)} - \widehat{\tau}^2 a_p^{(2)} + 2\mu_1 a_p^{(2)} - 2\mu_2 a_p^{(2)} - \widehat{\tau}^2 a_p^{(1)} + 2\mu_2 a_p^{(1)} \\ &= -\widehat{\tau}^2 (a_p^{(1)} + a_p^{(2)}) - 2a_p^{(1)} (\mu_1 - \mu_2) + 2a_p^{(2)} (\mu_1 - \mu_2) \\ &= -\widehat{\tau}^2 (a_p^{(1)} + a_p^{(2)}) - 2(a_p^{(1)} - a_p^{(2)}) (\mu_1 - \mu_2). \end{aligned}$$

Next

$$\begin{aligned} t_{21} &= -2\mu_1 \chi^{(1)} + 2\mu_1 a_s^{(1)} a_p^{(2)} - 2\mu_2 a_p^{(2)} a_s^{(1)} + 2\mu_2 \chi^{(2)} \\ &= -\widehat{\tau}^2 + 2\mu_1 + 2\mu_1 a_s^{(1)} a_p^{(2)} - 2\mu_2 a_p^{(2)} a_s^{(1)} + \widehat{\tau}^2 - 2\mu_2 \\ &= 2(\mu_1 - \mu_2) + 2a_s^{(1)} a_p^{(2)} (\mu_1 - \mu_2) \\ &= (2 + 2a_s^{(1)} a_p^{(2)}) (\mu_1 - \mu_2). \end{aligned}$$

Next

$$\begin{aligned} t_{12} &= -2\mu_1 a_p^{(1)} a_s^{(2)} + 2\mu_1 \chi^{(1)} - 2\mu_2 \chi^{(2)} + 2\mu_2 a_p^{(1)} a_s^{(2)} \\ &= -2\mu_1 a_p^{(1)} a_s^{(2)} + \widehat{\tau}^2 - 2\mu_1 - \widehat{\tau}^2 + 2\mu_2 + 2\mu_2 a_p^{(1)} a_s^{(2)} \\ &= -2(\mu_1 - \mu_2) - 2a_p^{(1)} a_s^{(2)} (\mu_1 - \mu_2) \\ &= -(2 + 2a_p^{(1)} a_s^{(2)}) (\mu_1 - \mu_2). \end{aligned}$$

Then

$$\begin{aligned} t_{22} &= -2\mu_1 \chi^{(1)} a_s^{(2)} - 2\mu_1 a_s^{(1)} - 2\mu_2 \chi^{(2)} a_s^{(1)} - 2\mu_2 a_s^{(2)} \\ &= -\widehat{\tau}^2 a_s^{(2)} + 2\mu_1 a_s^{(2)} - 2\mu_1 a_s^{(1)} - \widehat{\tau}^2 a_s^{(1)} + 2\mu_2 a_s^{(1)} - 2\mu_2 a_s^{(2)} \\ &= -\widehat{\tau}^2 (a_s^{(1)} + a_s^{(2)}) + 2a_s^{(2)} (\mu_1 - \mu_2) - 2a_s^{(1)} (\mu_1 - \mu_2) \\ &= -\widehat{\tau}^2 (a_s^{(1)} + a_s^{(2)}) - 2(a_s^{(1)} - a_s^{(2)}) (\mu_1 - \mu_2). \end{aligned}$$

It is worth noting here that t_{21} and t_{12} vanish when the parameters are equal, while the other two terms do not. This just means there is transmission of the P and S waves with no mode conversions, as to be expected when there are no interfaces.

So the determinant of this 2×2 minor is

$$\begin{aligned} \det &= t_{11} t_{22} - t_{21} t_{12} \\ t_{11} t_{22} &= \widehat{\tau}^4 (a_s^{(1)} + a_s^{(2)}) (a_p^{(1)} + a_p^{(2)}) \\ &\quad + 2\widehat{\tau}^2 (a_s^{(1)} + a_s^{(2)}) (a_p^{(1)} - a_p^{(2)}) (\mu_1 - \mu_2) + 2\widehat{\tau} (a_p^{(1)} + a_p^{(2)}) (a_s^{(1)} - a_s^{(2)}) (\mu_1 - \mu_2) \\ &\quad + 4(a_s^{(1)} - a_s^{(2)}) (a_p^{(1)} - a_p^{(2)}) (\mu_1 - \mu_2)^2 \\ &= \widehat{\tau}^4 (a_s^{(1)} + a_s^{(2)}) (a_p^{(1)} + a_p^{(2)}) \\ &\quad + 4\widehat{\tau}^2 (a_p^{(1)} a_s^{(1)} - a_p^{(2)} a_s^{(2)}) (\mu_1 - \mu_2) \\ &\quad + 4(a_s^{(1)} - a_s^{(2)}) (a_p^{(1)} - a_p^{(2)}) (\mu_1 - \mu_2)^2 \\ &= \widehat{\tau}^4 (a_s^{(1)} a_p^{(2)} + a_s^{(2)} a_p^{(1)}) + \widehat{\tau}^4 (a_s^{(1)} a_p^{(1)} + a_s^{(2)} a_p^{(2)}) \\ &\quad + (\widehat{\tau}^2 + 2(\mu_1 - \mu_2))^2 a_p^{(1)} a_s^{(1)} + (\widehat{\tau}^2 - 2(\mu_1 - \mu_2))^2 a_p^{(2)} a_s^{(2)} \\ &\quad - \widehat{\tau}^4 a_p^{(1)} a_s^{(1)} - 4a_p^{(1)} a_s^{(1)} (\mu_1 - \mu_2)^2 - \widehat{\tau}^4 a_p^{(2)} a_s^{(2)} - 4a_p^{(2)} a_s^{(2)} (\mu_1 - \mu_2)^2 \end{aligned}$$

$$\begin{aligned}
 &+ 4(a_S^{(1)} - a_S^{(2)})(a_P^{(1)} - a_P^{(2)})(\mu_1 - \mu_2)^2 \\
 &= \widehat{\tau}^4(a_S^{(1)}a_P^{(2)} + a_S^{(2)}a_P^{(1)}) \\
 &+ (\widehat{\tau}^2 + 2(\mu_1 - \mu_2))^2 a_P^{(1)}a_S^{(1)} + (\widehat{\tau}^2 - 2(\mu_1 - \mu_2))^2 a_P^{(2)}a_S^{(2)} \\
 &- 4a_P^{(1)}a_S^{(1)}(\mu_1 - \mu_2)^2 - 4a_P^{(2)}a_S^{(2)}(\mu_1 - \mu_2)^2 \\
 &+ 4(a_S^{(1)} - a_S^{(2)})(a_P^{(1)} - a_P^{(2)})(\mu_1 - \mu_2)^2 \\
 &= \widehat{\tau}^4(a_S^{(1)}a_P^{(2)} + a_S^{(2)}a_P^{(1)}) \\
 &+ (\widehat{\tau}^2 + 2(\mu_1 - \mu_2))^2 a_P^{(1)}a_S^{(1)} + (\widehat{\tau}^2 - 2(\mu_1 - \mu_2))^2 a_P^{(2)}a_S^{(2)} \\
 &- 4a_P^{(1)}a_S^{(1)}(\mu_1 - \mu_2)^2 - 4a_P^{(2)}a_S^{(2)}(\mu_1 - \mu_2)^2 \\
 &+ 4(a_P^{(1)}a_S^{(1)} + a_P^{(2)}a_S^{(2)})(\mu_1 - \mu_2)^2 - 4(a_S^{(1)}a_P^{(2)} + a_S^{(2)}a_P^{(1)})(\mu_1 - \mu_2)^2 \\
 &= \widehat{\tau}^4(a_S^{(1)}a_P^{(2)} + a_S^{(2)}a_P^{(1)}) \\
 &+ (\widehat{\tau}^2 + 2(\mu_1 - \mu_2))^2 a_P^{(1)}a_S^{(1)} + (\widehat{\tau}^2 - 2(\mu_1 - \mu_2))^2 a_P^{(2)}a_S^{(2)} \\
 &- 4(a_S^{(1)}a_P^{(2)} + a_S^{(2)}a_P^{(1)})(\mu_1 - \mu_2)^2
 \end{aligned}$$

Next, we have

$$\begin{aligned}
 -t_{21}t_{12} &= (2 + 2a_P^{(1)}a_S^{(2)})(2 + 2a_P^{(2)}a_S^{(1)})(\mu_1 - \mu_2)^2 \\
 &= 4(1 + a_P^{(1)}a_S^{(2)})(1 + a_P^{(2)}a_S^{(1)})(\mu_1 - \mu_2)^2 + 4(a_S^{(1)}a_P^{(2)} + a_S^{(2)}a_P^{(1)})(\mu_1 - \mu_2)^2
 \end{aligned}$$

Thus, after canceling the relevant terms, we obtain a nonzero determinant as long as $a_P^{(j)}$ and $a_S^{(j)}$ are not all complex:

$$\begin{aligned}
 \det &= \widehat{\tau}^4(a_S^{(1)}a_P^{(2)} + a_S^{(2)}a_P^{(1)}) \\
 &+ (\widehat{\tau}^2 + 2(\mu_1 - \mu_2))^2 a_P^{(1)}a_S^{(1)} + (\widehat{\tau}^2 - 2(\mu_1 - \mu_2))^2 a_P^{(2)}a_S^{(2)} \\
 &+ 4(1 + a_P^{(1)}a_S^{(2)} + a_P^{(2)}a_S^{(1)})(\mu_1 - \mu_2)^2.
 \end{aligned}$$

Next, notice that $t_{13}, t_{23}, t_{31}, t_{32} = 0$. We may also calculate

$$t_{33} = -\mu^{(2)}a_S^{(2)} - \mu^{(1)}a_S^{(1)} \neq 0$$

away from glancing and this concludes our proof that T is microlocally invertible in the microlocally nonelliptic region. □

A.2. Proof of Lemma 5.8

We can now do the tedious computation required to prove Lemma 5.8, which states that one may recover the infinitesimal jumps in wave speeds from the reflection operator.

Proof of Lemma 5.8. We would like to compute $R = Q_{21}T$ as well, or at the least check that it is invertible. As before, it suffices to check $[(S^{(1)})^T KE^{-1}](ES^{(2)})_{21}$ is invertible. Then the first 2×2 minor of this matrix is given by the multiplication of

$$\begin{bmatrix} 2\mu_1 a_P^{(1)} & 2\mu_1 \chi^{(1)} & 1 & a_P^{(1)} \\ -2\mu_1 \chi^{(1)} & 2\mu_1 a_S^{(1)} & -a_S^{(1)} & 1 \end{bmatrix} \begin{bmatrix} 1 & a_S^{(2)} \\ -a_P^{(2)} & 1 \\ -2\mu_2 a_P^{(2)} & -2\mu_2 \chi^{(2)} \\ 2\mu_2 \chi^{(2)} & -2\mu_2 a_S^{(2)} \end{bmatrix} = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix}.$$

First, we have

$$\begin{aligned}
 z_{11} &= 2\mu_1 a_P^{(1)} - 2\mu_1 \chi^{(1)} a_P^{(2)} - 2\mu_2 a_P^{(2)} + 2\mu_2 a_P^{(1)} \chi^{(2)} \\
 &= 2\mu_1 a_P^{(1)} - a_P^{(2)}(\widehat{\tau}^2 - 2\mu_1) - 2\mu_2 a_P^{(2)} + a_P^{(1)}(\widehat{\tau}^2 - 2\mu_2) \\
 &= \widehat{\tau}^2(a_P^{(1)} - a_P^{(2)}) + 2a_P^{(1)}(\mu_1 - \mu_2) + 2a_P^{(2)}(\mu_1 - \mu_2) \\
 &= \widehat{\tau}^2(a_P^{(1)} - a_P^{(2)}) + 2(a_P^{(1)} + a_P^{(2)})(\mu_1 - \mu_2).
 \end{aligned}$$

Next, we have

$$\begin{aligned} z_{21} &= -2\mu_1\chi^{(1)} - 2\mu_1a_p^{(2)}a_s^{(1)} + 2\mu_2a_p^{(2)}a_s^{(1)} + 2\mu_2\chi^{(2)} \\ &= -(\widehat{\tau}^2 - 2\mu_1) - 2a_p^{(2)}a_s^{(1)}(\mu_1 - \mu_2) + (\widehat{\tau}^2 - 2\mu_2) \\ &= 2(\mu_1 - \mu_2) - 2a_p^{(2)}a_s^{(1)}(\mu_1 - \mu_2) \\ &= (2 - 2a_p^{(2)}a_s^{(1)})(\mu_1 - \mu_2). \end{aligned}$$

Continuing,

$$\begin{aligned} z_{12} &= 2\mu_1a_p^{(1)}a_s^{(2)} + 2\mu_1\chi^{(1)} - 2\mu_2\chi^{(2)} - 2\mu_2a_p^{(1)}a_s^{(2)} \\ &= 2a_p^{(1)}a_s^{(2)}(\mu_1 - \mu_2) - 2\mu_1 + 2\mu_2 \\ &= (-2 + 2a_p^{(1)}a_s^{(2)})(\mu_1 - \mu_2). \end{aligned}$$

Lastly,

$$\begin{aligned} z_{22} &= -2\mu_1a_s^{(2)}\chi^{(1)} + 2\mu_1a_s^{(1)} + 2\mu_2a_s^{(1)}\chi^{(2)} - 2\mu_2a_s^{(2)} \\ &= -a_s^{(2)}(\widehat{\tau}^2 - 2\mu_1) + 2\mu_1a_s^{(1)} + a_s^{(1)}(\widehat{\tau}^2 - 2\mu_2) - 2\mu_2a_s^{(2)} \\ &= \widehat{\tau}^2(a_s^{(1)} - a_s^{(2)}) + 2a_s^{(2)}(\mu_1 - \mu_2) + 2a_s^{(1)}(\mu_1 - \mu_2) \\ &= \widehat{\tau}^2(a_s^{(1)} - a_s^{(2)}) + 2(a_s^{(1)} + a_s^{(2)})(\mu_1 - \mu_2). \end{aligned}$$

We also have

$$z_{33} = -\mu^{(2)}a_s^{(2)} + \mu^{(1)}a_s^{(1)}$$

It will be convenient to denote $R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$ the individual entries. Next, notice that

$r_{13}, r_{23}, r_{31}, r_{32} = 0$ since the corresponding entries for T and Q_{21} are as well. Using the calculation for T , we may calculate

$$r_{33} = \frac{\mu^{(1)}a_s^{(1)} - \mu^{(2)}a_s^{(2)}}{\mu^{(1)}a_s^{(1)} + \mu^{(2)}a_s^{(2)}}.$$

We may then compute $(r_{33} - 1)/(r_{33} + 1) = (\mu^{(2)}a_s^{(2)})/(\mu^{(1)}a_s^{(1)})$, so since $\mu^{(1)}$ and $a_s^{(1)}$ is already determined, we recover $\mu^{(2)}a_s^{(2)} = \sqrt{\mu^{(2)}|\xi'|^2 - \tau^2}$. Since the tangential momenta ξ', τ are already determined, we recover $\mu^{(2)}(x_0)$ and thereby $a_s^{(2)}(x_0, \tau_0, \xi_0)$.

All we have left to determine is $a_p^{(2)}$ which would give us $\lambda^{(2)}(x_0)$. For this, we use the first 2×2 minor of T and Q_{21} , and after a tedious computation, since everything is known except $\lambda^{(2)}$, we get the recovery by similar arguments as above.

We assumed throughout these calculations that $|\xi'|$ lies away from zero. However, at 0, the calculations are much simpler and follow the same arguments. \square

B. Proof of Proposition 5.3

Before proving this proposition, it will be useful to have a notion of the *direct transmission* constituent of a wavefield, defined microlocally.

Microlocal almost direct transmission

As in [4, section 3], we are interested in isolating the *microlocal almost direct transmission* since this will be the main tool necessary to prove the main theorem in the presence of multiple scattering. Intuitively, it is the microlocal restriction of the solution at time T (say) to singularities in $T^*\Theta$ whose P -distance from the surface $\partial T^*\Theta$ is at least T . Formally, first let $(T^*M)_t$ be the set

of covectors of depth greater than t in a manifold M :

$$(T^*M)_t = \{\zeta \in T^*M \mid d_{T^*M}^*(\zeta) > t\}$$

where $d_{T^*M}^*(\zeta)$ is defined as in [7, section 3] using d_P .

Instead, we consider smooth cutoffs, and for choose nested open sets Θ', Θ'' between Ω and Θ :

$$\Omega \subset \Theta' \subset \overline{\Theta'} \subset \Theta'' \subset \overline{\Theta''} \subset \Theta.$$

A *microlocal almost direct transmission* of h_0 at time T is a distribution h_{MDT} satisfying

$$h_{\text{MDT}} \equiv R_T h_0 \quad \text{on } (T^*\Theta)_T \quad \text{WF}(h_{\text{MDT}}) \subset \overline{(T^*\Theta'')_T}.$$

We present the more precise definition of the directly transmitted component of the forward elastic wave propagator R_T^+ .

Definition B.1. (The microlocal direct transmission). Fix $(x, \zeta) \in T^*\Omega$ and let $\gamma_{P/S}$ be a purely transmitted P or S ray, starting outside Ω at $t=0$ and passing through (x, ζ) at some time $t=T$. Assume $\gamma_{P/S}$ intersects Γ exactly k times. Define the *directly transmitted wave constituent* FIO

$$\mathbf{DT}_{k,P/S}^+ = \begin{cases} r_T \Pi_{P/S} J_{C \rightarrow S}, & k = 0, \\ r_T \Pi_{P/S} J_{\partial \rightarrow S} t M_T (J_{\partial \rightarrow \partial} \Pi_{P/S} t M_T)^{k-1} J_{S \rightarrow \partial} \Pi_{P/S} J_{C \rightarrow S}, & k > 0 \end{cases} \quad (\text{B.1})$$

By construction, if h_0 is a distribution of Cauchy data with $\text{WF}(h_0) = \mathbb{R}_+ \gamma_{P/S}(0)$, then $\mathbf{DT}_{k,P/S}^+ h_0$ will be a distribution whose wavefront set is equal to $(x, \mathbb{R}_+ \zeta)$. Moreover, it will be a polarized P/S wave. At each point in time, the direct transmission is a microlocal restriction of the wavefield which has wavefront set contained in a small neighborhood of a purely transmitted P or S bicharacteristic.

The key to prove this proposition is to isolate h_{MDT} . For this, we need access outside of Ω to all scattered rays related to h_{MDT} (*unutilized* bicharacteristics defined below), which certainly includes all possible S -rays as well.

Definition B.2.

- a. Let $\gamma : [0, t^*] \rightarrow T^*\mathbb{R}^3$ be a purely transmitted, broken, P or S ray that starts outside Ω and hits k interfaces at discrete times $t_1, \dots, t_k > 0$ say. At such a time t_i , γ will have one or two reflecting branches and one possible mode converted branch according to Snell's law. Such branches that are reflections are called *unutilized reflecting rays* associated to γ . A transmitted branch that is a mode conversion associated to γ will be called a *unutilized transmission*. Essentially, for a purely transmitted ray, γ , we seek to use in our proofs to recover local interior travel times, when γ hits an interface, the outgoing reflected branches and the mode converted transmitted branch (when it exists) are unutilized rays. The rays we eliminate with scattering control are either the unutilized transmissions or reflected rays produced from an unutilized reflection hitting an interface. These two types of rays we eliminate are precisely the ones that go deeper into the interior and interfere with our travel time measurements. A wave associated with such a ray will be called a *unutilized wave* and we may refer to either branch as a *unutilized ray* or *unutilized branch*. For the proof of the proposition, such unutilized reflecting waves are precisely the ones that will create waves (upon their next interaction with an interface) that need to be eliminated and will not be utilized to probe deeper into the medium. We must also ensure that these unutilized mode-conversion transmissions are eliminated as well.
- b. Bicharacteristics γ_1, γ_2 are *connected* if their concatenation $\gamma_1 \cup \gamma_2$ is a broken bicharacteristic. Note that mode conversions are allowed (e.g., a P ray may be connected to an S ray) if their tangential momenta match. A bicharacteristic γ_1 terminating at an interface may have one or two (totally reflected with P/S mode conversions), or two, three, or four (reflected and transmitted) connecting bicharacteristics there. If γ_1 has a transmitted bicharacteristic, there exists an *opposite* bicharacteristic γ sharing γ_1 's connecting bicharacteristics. There can be up to two opposite bicharacteristics (one for P and one for S). Basically, for an outgoing

ray β , say, moving away from the interface, its opposite bicharacteristics are all incoming rays on the other side of the interface that are connected to β ; hence, any incoming wave singular along an opposite bicharacteristic associated with β will produce a transmitted wave singular along β . Note that γ_1 or γ_2 (or both) may be glancing at an interface (see [4] for definitions). If it is not, we say that it is *non-glancing*.

- c. Fix a large time $T_s > 0$ (see below). A bicharacteristic $\gamma : (t_-, t_+) \rightarrow T^*(\mathbb{R}^3 \setminus \Gamma)$ is (\pm) -*escapable* if either:
 - d. it has *escaped*: γ is defined at $T \pm T_s$ and $\gamma(T \pm T_s) \notin T^*\Theta$, or recursively, after only finitely many recursions, either
 - i. all of its connecting bicharacteristics at t_{\pm} are (\pm) -escapable and are non-glancing at the interface;
 - ii. all of its reflecting bicharacteristics (both P and S if they exist) are (\pm) escapable and non-glancing and both P, S opposite bicharacteristics are (\mp) escapable and non-glancing.

Remark B.3. The t_{\pm} are just the times at the two endpoints of a bicharacteristic segment γ . Thus, γ is merely a single branch of a family of broken bicharacteristics (determined via concatenated branches that we term reflections, transmissions, and mode conversions in the definition), in which $\gamma(t_{\pm})$ are the endpoints of the branch. The reason the (\pm) notation is being used is that for $(+)$ -escapability, say, we will follow a series of concatenated branches forward in time, continuing $\gamma(t_+)$ of each branch until it escapes. We are describing these escapable broken bicharacteristics recursively via their branches since the parametrix construction is simpler with a recursive formula. At each interface, we must consider both P (if present) and S transmissions—as well as any reflections, to ensure that unwanted multiple reflections can be controlled.

Let us explain the choice of T_s further. The idea is that we want a large enough time so that any returning bicharacteristic, even a concatenation of pure slow rays, will return to Θ^* by time T_s . In fact, there is a unique minimal time that has these properties. This will ensure that there is enough time for all wave constituents of a particular u_h associated to returning bicharacteristics eventually return to Θ^* by time T_s . This avoids the problems encountered in the lack of control in the previous appendix. We also note that T_s is not used to discriminate between certain rays and we can even allow it to vary for different rays since we are allowed infinite time to take exterior measurements of the wavefield. Rather, it just makes the notation less cumbersome to have a fixed time beyond which other rays are irrelevant and will not affect the construction of the “tail”.

The idea is that $(+)$ -escapable singularities are ones we do not worry about since they escape and do not enter the directly transmitted region. Once there are connecting rays that are not $(+)$ -escapable, then those need to be eliminated. Thus, property (c) iii. in Definition B.2 guarantees that corresponding to these non-escapable rays, there are corresponding *opposite* $(-)$ -escapable rays that we use to send in waves to eliminate waves associated to rays that do not escape. The previous definition of unutilized rays are exactly the ones that create (through geometric optics concatenation) the non-escapable rays just described.

In the final case, if the (\pm) -escapable connecting bicharacteristic is a reflection, then we require that there are both P and S opposite bicharacteristics that escape. They must be there so that we can construct an incoming wave parametrix, singular on such opposite rays, that eliminates a particular scattered wave. They must also escape so that we obtain all the necessary information not just from the fast moving P -waves, but the S -waves as well. This is because we must construct S waves in addition to P waves for the tail, even if one is merely trying to eliminate a single P wave.

To help illustrate these definitions, we refer the reader to [4, Figure 3.7] for a visualization in the acoustic setting, where in that setting, unutilized rays are called “returning.”

Remark B.4. The definition of escapability merely ensures that in our measurement region, we access all backscattered rays caused by the direct transmission. In addition, the recursive

definition and the notion of *opposite* rays ensures that for any *new* nonescapable scattered waves created by an appropriate tail, one will be able to eliminate them as well if they enter the direct transmission’s domain of influence. The conditions should be compared to the linear problem of obtaining an observability inequality with partial data. There as well, one needs access to all the relevant rays in the measurement region.

Remark B.5. Note that the definition of (\pm) -escapable rays can be made more general to deal with more general geometries that do not assume a convex foliation. Since we only need to eliminate waves associated to rays that are not escapable, this only requires having the associated number of opposite escapable rays. Thus, we need the number of opposite (\mp) -escapable rays to equal the number of not (\pm) -escapable rays that are reflecting or transmitting.

We recall $\mathcal{S} \subset T^*\Omega$ as the set of ξ such that every unutilized bicharacteristic belonging to a broken bicharacteristic through ξ is *(+)-escapable*. Note that the definition of escapable ensures that the mode conversion are non-glancing as well. For example, if a purely transmitted P bicharacteristic starts outside Ω and passes through ξ , then all the transmitted S , mode-converted connecting rays are non-glancing. An analog holds for a purely transmitted S bicharacteristic.

Lastly, before beginning the proof, we cannot construct our usual parametrices near glancing rays. Fortunately, the extended convex foliation condition will guarantee that “most” (in a sense to be described soon) broken bicharacteristics (that travel for a fixed finite time T say) will not glance at an interface.

Lemma B.6. *Let $\mathcal{G} = \mathcal{G}_T \subset T^*\Theta$ be the set of $(y, \eta) \in T^*\Theta$ such that a broken bicharacteristic passing through (y, η) of length T contains a glancing point. \mathcal{G} is a manifold and under the extended convex foliation assumption, it has dimension at most $2n - 2$. Thus, the set of covectors $(y, \eta) \in T^*\Theta$ where all broken rays of length T passing through (y, η) never glance is dense in $T^*\Theta$.*

We now present a series of lemmas to demonstrate that under the extended convex foliation condition, we have enough covectors lying in \mathcal{S} .

Lemma B.7. *Let γ be a transmitted geodesic with respect to some wave speed c . Then $\rho \circ \gamma$ either monotonically decreases, strictly monotonically increases, or strictly decreases then strictly increases.*

Proof. Suppose, on the contrary, that $\rho \circ \gamma$ is nondecreasing on $[a, b]$ then nonincreasing on $[b, d]$ for some $a < b < d$. Let $\tau = \rho(b)$. If c is smooth near $\gamma(b)$ then there is a neighborhood $(a', d') \subset [a, d]$ of b such that $\rho(\gamma(a')) = \rho(\gamma(d')) = \tau' \geq \tau$. Then $\gamma|_{[a', d']}$ is a geodesic between points on $\Sigma_{\tau'}$ outside of $\Omega_{\tau'}$, contradicting the strict convexity of $\partial\Omega_{\tau'}$. Conversely, if c is discontinuous at $\gamma(b)$, then $\gamma((a, b))$ and $\gamma((b, d))$ are on opposite sides of Γ , which is locally given by Σ_τ , by the definition of a transmitted geodesic. This is a contradiction as well. □

The next lemma states that upward-traveling geodesics are not trapped.

Lemma B.8. *The set of $(x, \xi) \in T^*_-\Omega$ for which there exists a purely transmitted geodesic $\gamma : [a, b] \rightarrow \bar{\Omega}$ with $\gamma'(0)^b = (x, \xi)$ and $\gamma(a), \gamma(b) \in \partial\Omega$ is open and dense in $T^*_-\Omega$.*

Proof. Our restriction to foliation upward covectors is needed to avoid total internal reflections, which would prevent γ from reaching the boundary.

By symmetry, it suffices to show that we can find γ with one endpoint, say b , on $\partial\Omega$. Let $\gamma : T \rightarrow T^*\mathbb{R}^3$ be the unique maximal purely transmitted bicharacteristic with $\gamma(0) = (x, \xi)$, and let γ be its (continuous) projection onto \mathbb{R}^3 . If $\gamma(b) \in \partial T^*\Omega$ we are done, so assume this does not hold, and let $s = \sup I$.

If $s < \infty$, then $\gamma(s) \in \Gamma$ since otherwise the geodesic could be continued. There are two possibilities: γ glances off Γ ($\gamma(s) \in T^*\Gamma$), or there is total internal reflection. In the first case, note

that γ is in the flowout of $T^*\Gamma \setminus 0$ under Φ ; this has measure zero in $T^*\Omega$ because Φ is piecewise smooth and $\dim T^*\Gamma = \dim T^*\Omega - 2$.

In the second case, c is smaller on the side of Γ opposite $\gamma(s^-)$ by B.7 convexity of the interfaces, as noted in the proof of [Theorem 1.1](#). This rules out internal reflection, so γ can be continued past s , a contradiction.

Let us suppose now that $s = \infty$. By lemma B.7, $\rho \circ \gamma$ is increasing on $(0, \infty)$. Let $\rho^* = \inf_\gamma \rho$, and choose a sequence $s_j \rightarrow \infty$ such that $\rho(\gamma(s_j)) \rightarrow \rho^*$. By compactness, s_j has a subsequence (which we may again label s_j) such that $\gamma(s_j)$ converges to some point $(x, \xi) \in T^*\Omega$, and by continuity $(x, \xi) \notin T^*_+\Omega$. However, by strict convexity the geodesic starting at any $(x, \xi) \in T^*_+\Omega$ immediately leaves $T^*\Omega \setminus T^*_\tau\Omega$. This is true even if $x \in \Gamma$. By continuity, this is true if we replace (x, ξ) by any sufficiently close covector and in particular $\gamma(s_j)$ for sufficiently large j (as noted above, total internal reflection cannot occur). Hence ρ^* cannot be the infimum of ρ on γ , a contradiction. \square

The density allows us to just recover the wave speeds at points where all possible rays through the point never glance.

Proof of Proposition 5.3 Because any broken ray intersects only finitely many interfaces in the time interval $t \in [0, 2T]$, the condition of being (\pm) -escapable is open, and in particular \mathcal{S} is open.

Construction of h_0

Let γ be a purely transmitted P -bicharacteristic starting at $\partial\Theta$ (when projected to the base space) for $t=0$ and $\gamma(T) = (x, \xi)$. Let h_0 be any Cauchy data supported in $\Theta \setminus \Omega$ with wavefront set containing $\mathbb{R}_+\gamma(0)$ so that inside Θ_T , $\text{WF}(R_T h_0) = \text{WF}(v)$ by finite speed of propagation (see [Figure 2](#)). This would actually suffice for our purposes, but since we have the FIO calculus, we can construct h_0 more carefully for the improved result that $\text{WF}(R_T h_0 - v) \cap T^*\bar{\Omega}_T = \emptyset$.

Let us give a brief argument on one possible construction of such h_0 described above. First define $J_{\partial \rightarrow \partial}^- = \nu J_{\partial \rightarrow \partial} \nu$ (ν is the time reversal operator defined in Section 2.1), which is like $J_{\partial \rightarrow \partial}$ but propagating backwards in time. Let ν be distribution with wavefront set $\mathbb{R}(x, \xi)$ and let d be the number of interfaces between x and Ω^c . Then define $h_0 = J_{C \rightarrow \partial}^{-1} M_T^{-1} \prod_{i=1}^d (J_{\partial \rightarrow \partial}^- M_T^{-1}) \nu|_\Gamma$. The wavefront set of h_0 (viewed in the cosphere bundle) will consists of up to 2^d covectors (see [Figure 2](#)).

The key now is to construct a tail that eliminates the multiple scattering and justify that such an h_0 above can be constructed.

Construction of K_{tail}

We first define FIO's $\Xi_\pm^I, \Xi_\pm^O : \mathcal{C}^\infty(\mathbb{R} \times \partial Z) \rightarrow \mathcal{D}'(Z)$ of order 0 producing tails outside Θ for (\pm) -escapable bicharacteristics. The $\Xi_\pm^{I/O}$ -constructed tail for a singularity on a $(+)$ -escapable bicharacteristic ensures this singularity escapes Θ by time $T + T_s$, without generating any singularities in h_{MDT} 's P/S -domain of influence where h_{MDT} is associated to a purely transmitted P/S -ray. The $\Xi_\pm^{I/O}$ -constructed tail generates a given singularity on a $(-)$ -escapable bicharacteristic, again without causing any singularities in the P/S domain of influence. The Ξ_\pm^O are defined on outgoing boundary data while the Ξ_\pm^I are defined on incoming data, microlocally near the final, resp., initial covectors of (\pm) -escapable bicharacteristics.

Let $\gamma : (t_-, t_+) \rightarrow T^*Z$ be (\pm) -escapable bicharacteristic. Denote by β^O the pullback to the boundary of its final point: $\beta^O = (d\iota_\Gamma)^* \gamma(t_+)$, where by abuse of notation we consider $\gamma(t_\pm)$ as a space-time covector $T^*(\mathbb{R} \times Z)$. Define $\beta^I = (d\iota_\Gamma)^* \gamma(t_-)$ similarly. We now define $\Xi_\pm^{I/O}$ microlocally near $\beta^{I/O}$, starting with the incoming maps Ξ_\pm^I .

- If $t_\pm \in (0, T + T_s)$: We simply follow the bicharacteristic and apply Ξ_\pm^O at the other end. In the $(+)$ case define $\Xi_+^I \equiv \Xi_+^O J_{\partial \rightarrow \partial}$. In the $(-)$ case, define $\Xi_-^I \equiv \Xi_-^O J_{\partial \rightarrow \partial}^-$ near β^I , where $J_{\partial \rightarrow \partial}^- = \nu J_{\partial \rightarrow \partial} \nu$ is like $J_{\partial \rightarrow \partial}$ but propagating backwards in time.

- If γ escapes, $t_{\pm} \notin [0, T + T_s]$: This is the terminal case. In the (+) case, there is nothing to do: define $\Xi_{\pm} \equiv 0$ near β^I . For the (-) case, define $\Xi_{-}^I \equiv J_{C \rightarrow \partial}^{-1}$ near β^I to obtain the necessary Cauchy data,

We now turn to Ξ_{\pm}^O , considering each of the cases of in the definition of (\pm) -escapability.

- if γ escapes: This case never arises: Ξ_{\pm}^I is not defined in terms of Ξ_{\pm}^O for such γ .
- If all outgoing bicharacteristics are (\pm) -escapable: Recursively apply Ξ_{\pm}^I to the reflected and transmitted (if any) bicharacteristics, defining $\Xi_{\pm}^O \equiv \Xi_{\pm}^I M$ near β^O .
- If all the reflecting bicharacteristics are (\pm) -escapable, and the opposite incoming P/S rays are (\mp) -escapable: This is the core case. In the (+) case, near β^O let

$$\Xi_{+}^O \equiv -\Xi_{-}^I M_T^{-1} M_R + \Xi_{+}^I (M_T - M_R M_T^{-1} M_R),$$

The inverses here are all microlocal near the appropriate covector. The (-) case is slightly different: near β^O ,

$$\Xi_{-}^O \equiv -\Xi_{-}^I M_T^{-1} + \Xi_{+}^I M_R M_T^{-1}.$$

Given $\eta \in \mathcal{S} \subset T^* \Theta^o$, consider all the unutilized reflecting, (+)-escapable rays associated to η . Each is associated with a distinct sequence of reflections and transmissions $a = (a_1, \dots, a_k) \in \{R, T\}^k$ for some k and corresponding P/S wave microlocal mode projections $\Pi_{\lambda_j}, \lambda = (\lambda_1, \dots, \lambda_k)$, and a corresponding propagation operator

$$\mathcal{P}_{a, \lambda, R} = J_{\partial \rightarrow \partial} \Pi_{\lambda_k} M_{a_k} \cdots J_{\partial \rightarrow \partial} \Pi_{\lambda_2} M_{a_2} J_{\partial \rightarrow \partial} \Pi_{\lambda_1} M_{a_1} J_{C \rightarrow \partial}.$$

Notice the λ is here so that we are observing the wave (with possible reflection, transmission, and mode conversions) associated to a single broken bicharacteristic consisting of a concatenation of P and S rays. Likewise we can define $\mathcal{P}_{a, \lambda, T}$ for the transmitting, unutilized rays that are (-)-escapable. These transmitting unutilized rays are new for the systems setting due to multiple wave speeds and were not present in the acoustic setting of [4]. Let $\mathfrak{G}_{\eta}^{+} = \mathfrak{G}_{\eta}^{+}$ be the set of all such finite sequences (a, λ) associated to unutilized reflecting, (+)-escapable rays associated to η . Likewise, $\mathfrak{G}_{\eta}^{-} = \mathfrak{G}_{\eta}^{-}$ is the set of such (a, λ) associated to the transmitting, unutilized rays that are minus escapable and associated to η .

First define

$$A_{\eta} = \Xi_{+}^O \sum_{(a, \lambda) \in \mathfrak{G}^{+}} \mathcal{P}_{a, \lambda, R} + \Xi_{-}^O \sum_{(a, \lambda) \in \mathfrak{G}^{-}} \mathcal{P}_{a, \lambda, T},$$

and then define A by patching together the A_{η} with a microlocal partition of unity as in [4]. Given an h_0 , the tail is precisely

$$K_{\text{tail}} := Ah_0$$

The remainder of the proof follows simply by construction of Ξ_{\pm}^O and Ξ_{\pm}^I . Recall our construction that inside $T^* \Omega_{\tau}$, $\text{WF}(R_T h_0) = \text{WF}(v)$ for some large enough T . One just needs T to be greater than the P or S (depends on which case in the proposition we are considering) distance between (x, ξ) and $S^* \Omega^c$, and one can increase T after that by adjusting h_0 . Set $h_{\infty} = h_0 + K_{\text{tail}}$, and we must verify that $Fh_{\infty}|_{\Omega_{\tau}} \equiv v$ for $t \geq T_s$. Any other waves in this region may only come from $\mathcal{P}_{a, \lambda, T} h_0$ or $R_t \Xi_{\pm}^O \mathcal{P}_{a, \lambda, R/T} h_0$ for some t and $(a, \lambda) \in \mathfrak{G}^{\pm}$. But by construction of Ξ_{\pm}^O , any such unutilized wave from $\mathcal{P}_{a, \lambda, T} h_0$ will get canceled by $\Xi_{\pm}^O \mathcal{P}_{a, \lambda, R/T} h_0$. The recursive definition also ensures that any new unutilized wave created by $\Xi_{\pm}^O \mathcal{P}_{a, \lambda}^{(t)} h_0$ also gets eliminated. Thus, neither of these constituents may produce waves whose singularities enter Ω_{τ} microlocally and that completes the proof. □