

Recovery of Piecewise Smooth Density and Lamé Parameters from High Frequency Exterior Cauchy Data*

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Abstract. We consider an isotropic elastic medium occupying a bounded domain $\Omega \subset \mathbb{R}^3$ whose density and Lamé parameters are piecewise smooth. In the elastic wave initial value inverse problem, we are given the solution operator for the elastic wave equation, but only outside Ω and only for initial data supported outside Ω , and we study the recovery of the density and Lamé parameters. For known density, results have recently been obtained using the scattering control method to recover wave speeds. Here, we extend this result to include the recovery of the density in addition to the Lamé parameters under certain geometric conditions using techniques from microlocal analysis and a connection to local tensor tomography.

Key words. inverse problems, elastic wave equation, microlocal analysis

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1. Introduction. The main goal of this work is to recover a piecewise smooth density of mass in addition to the other elastic parameters in an isotropic elastic setting using exterior measurements. In general, the wave inverse problem asks for the unknown coefficient(s), representing wave speeds, of a wave equation inside a domain of interest $\Omega \subset \mathbb{R}^3$, given knowledge about the equation's solutions (typically on $\partial\Omega$). Consider the isotropic elastic wave equation in a bounded domain $\Omega \subset \mathbb{R}^3$ with smooth boundary. The wave operator for elastodynamics is given as

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$$(1.1) \quad \mathcal{P} = \rho \partial_t^2 - W$$

with

$$W = \nabla \cdot (\lambda \operatorname{div} \otimes \operatorname{Id} + 2\mu \widehat{\nabla}),$$

where ρ is the density of mass that we term simply as *density*, λ and μ are the Lamé parameters, and $\widehat{\nabla}$ is the *symmetric gradient* used to define the strain tensor for an elastic system via $\widehat{\nabla}u = (\nabla u + (\nabla u)^T)/2$ for a vector valued function u . Operator \mathcal{P} acts on a vector valued distribution $u = (u_1, u_2, u_3)$, the *displacement* of the elastic object. Let $\Gamma_1, \dots, \Gamma_k$ be closed disjoint, smooth surfaces (interfaces) splitting Ω into subdomains, each with an exterior boundary given by some Γ_i and some interior boundary Γ_j ; see [28, Figure 1, left] for an illustrative example. In this work, we assume that the ρ , λ , and μ are functions, smooth (C^∞) up to these interfaces with possible jumps there. That is, ρ, λ, μ have conormal singularities at the interfaces.

The motivation of this inverse problem comes from seismic imaging of the Earth where the density and the Lamé parameters typically jump across geological interfaces within the crust, and across major discontinuities associated with phase transitions in the deep interior, but the methods we employ also have engineering applications in nondestructive testing of elastic materials. As explained in [23], there is a long history of using high frequency waves to detect flaws in structural materials, and by understanding the scattering of elastic waves by flaws in an otherwise isotropic elastic space, one can solve an inverse problem to detect the flaws. In our problem, we use welded contact transmission conditions which preclude flaws. However, the principles of our approach could be adapted (in the acoustic case) to the presence of flaws nonetheless with applications to nondestructive testing as in [23]. In our inverse problem we only consider dynamic moduli since our inverse problem is with waves, and we are mainly focused on the density of mass. It is interesting to note that in engineering, outside of highly engineered materials (steel, carbon-based composites, etc.), density and elastic (Lamé) properties are often not correlated. In our inverse problem as well, we do not make any assumptions about a relation between density and Lamé parameters.

Traditionally, the coefficients of the elastic system are smooth, and the data is the Dirichlet-to-Neumann map, or its inverse. The main questions are uniqueness and stability: Can the coefficients be recovered from the Dirichlet-to-Neumann map, and is this reconstruction stable relative to perturbations in the data? In the case of a scalar wave equation with smooth coefficients, a number of results such as [2, 30, 24] have answered the question in the affirmative. For the piecewise smooth case, a novel scattering control method was developed in [7] in order to show in [5] that uniqueness holds as well for piecewise smooth wave speeds with conormal singularities, under mild geometric conditions. Less is known in the elastic setting as will be described, but several works such as [13, 32, 28] show how to construct a Fourier integral operator (FIO) representation of the solution to the elastic wave equation near an interface, which is useful for inverse problems in the hyperbolic elastic setting where coefficients have conormal singularities. An additional challenge of recovering a coefficient that is not in the principal symbol of the operator is that one needs to solve a tensor tomography problem at some stage of the argument, which has a gauge freedom that obstructs uniqueness [18]. As will be explained, the gauge freedom in our case actually gets exploited to derive an elliptic equation that allows a unique recovery of the density.

In this work, we recover a piecewise smooth density of mass (in the isotropic elastic wave equation) in the interior of the domain, thereby recovering all three Lamé parameters. We essentially want to “image” the density using high frequency waves; the first step is a uniqueness result that we consider in this manuscript, but a partial reconstruction will be future work since much of the proof here is constructive. In [28], Stefanov, Uhlmann, and Vasy recover piecewise smooth wavespeeds from the Dirichlet-to-Neumann map under certain geometric conditions. Their argument relies on the principal symbol of the elastic operator and the parametrix. As noted in [28, Remark 10.2], their argument does not address unique determination of the density of mass past the first interface, nor at the interface itself. Since the density appears in the lower order part of the elastic operator, it is natural to look at the lower order symbols in the asymptotic expansion of elastic wave solutions. This leads to the inverse problem considered here where we study the lower order constituents of a parametrix, as done by Rachele in [22] in the smooth case, to recover a piecewise smooth density of mass.

For the isotropic, elastic setting with smooth parameters, the uniqueness question was settled by Rachele in [21] and Hansen and Uhlmann [14]. In [6], the authors extended these results to the isotropic elastic system, where the parameters are piecewise smooth. The main difficulty in the elastic system setting is lack of the sharp form of the unique continuation result of Tataru [29] since one has to deal with two different wave speeds, whereas in the scalar wave setting, the wave speed may be reconstructed without the microlocal approach we take here by relying on a unique continuation result (see [5]). The main result of [6] is that under a *convex foliation condition* on the domain, one can uniquely determination the P/S -wave speeds that contain singularities via microlocal analysis, scattering control, and a layer stripping argument akin to [25]. In [6], it is assumed that the density ρ was trivial, but with similar yet more sophisticated arguments, it was proved in [28] that this assumption can be dispensed with, and that both piecewise smooth wave speeds can be recovered from exterior measurements even when the density is piecewise smooth. The simpler case of piecewise analytic and piecewise constant coefficients is considered in [9, 8] and the arguments are quite different from our approach here. In our approach, low frequencies are not required in the data.

Recovering the material density does not simply follow from the arguments in [28, 6]. This is because those arguments only rely on the principal symbol of the elastic operator and the principal term in the high frequency asymptotic expansion of solutions to the elastic wave equation to recover travel times, and the principal symbol contains no information about the density [21, 22]. Rachele also showed in [22] that the polarization of the waves does not contain information about the density. By looking at the lower order terms of the amplitude of an FIO representation of an elastic wave solution, Rachele shows [22] that in the smooth setting, one may recover the density as well under certain conditions. That was a global result, but by using the results of [25] in local tensor tomography, Battacharyya shows in [3] that one can locally recover the density as well near the boundary in the smooth setting. We aim to extend the results in [22] to the piecewise smooth setting and recover a piecewise smooth density of mass. We also note that Rachele’s results in [21, 3] assume that the manifold is simple so that there are no caustics. Here, we do not assume that the manifold is simple, but we do assume a “convex foliation condition” that will be described shortly, which allows for caustics and, in particular, is satisfied by manifolds with nonnegative curvature. Another novelty is that previous papers [20, 22] used plane waves to recover parameters and lower

order amplitude data that reduced the inverse problem to a tensor tomography problem. In the smooth setting, the Dirichlet-to-Neumann map applied to a plane wave is a Lagrangian distribution corresponding to a single Lagrangian, where one may “peel off” each term in the asymptotic expansion of its symbol. This approach is not known to work to recover such data past an interface since due to the multiple scattering, the measured data using an initial plane wave would be a complicated sum of Lagrangian distributions that are difficult to disentangle. Thus, we use more general distributions and show how to recover “lower order polarizations” of elastic waves locally beyond an interface or multiple interfaces.

In Theorem 1.1 we show that under an *extended convex foliation condition* (see section 3) and outside a specific set, we can uniquely determine all three Lamé parameters that contain singularities. We first recover the wave speeds as in [28, 6] via microlocal analysis, scattering control, and a layer stripping argument akin to [25] by recovering and manipulating local travel time data. To recover ρ from the boundary up until the first interface, the results of [3] may be employed as shown in [28], where they recover all three parameters up until the first interface, but do not address recovering the density of mass past the first interface. To recover the density (and all its derivatives) across the interface, we first send in high frequency waves at the interface and measure their reflected amplitudes (we can do this since we have already recovered the full elastic operator above this interface) to recover the reflection operator at the interface as in [6]. The reflection operator is a classical 0th order pseudodifferential operator (PsiDO), and we recover all the terms in the polyhomogeneous expansion of its symbol. We may then use the result in [4] to recover the jet of ρ and all its derivatives infinitesimally below the interface. We then send in high frequency waves that generate a transmitted P -wave directly below the interface, which then travels near the interface before returning to it. We measure the amplitude of this P -wave to obtain “lower order polarization data” which involves the density, and recovering the density gets reduced to a local 2-tensor tomography problem by a careful analysis of this lower order amplitude. Normally, this would create gauge freedom in the recovery of the density, but we employ the argument of [3, 22] to show that the gauge freedom actually leads to an elliptic equation for the density, where we can then locally recover the density in the interior below the interface. We proceed in this way until the density is recovered in all layers.

A subtle technical point is that we probe the medium with distributions that have wavefront at a single covector (modulo the \mathbb{R}^+ group action) in $T^*\Omega$ (this is the cotangent bundle of Ω), and hence, locally and away from any interface, the waves we generate have a wavefront set consisting of a single bicharacteristic. Such distributions do not fall into a specific Hörmander class of conormal distributions, and our given measurements have the form FV where F is a specific FIO with a polyhomogeneous symbol and V is just an arbitrary distribution. To obtain the “lower order polarization” amplitudes, we need to extract the principal symbol as well as the lower order symbols associated to F from such data. In section 4, we derive a variation of the Weinstein symbol calculus in [31] that applies to arbitrary distributions and extend those results to get a product-type formula for the principal symbol of an FIO applied to an arbitrary distribution. The principal term can then be “peeled off” to obtain the lower order amplitudes by a similar procedure.

Most proofs are microlocal to avoid using unique continuation results, but we require an important geometric assumption, which is an *extended convex foliation condition* (see section

3.2 for the smooth setting) for each wave speed $c_{P/S}$. As mentioned in [25], for a particular wave speed, this condition relates to the existence of a function with strictly convex level sets, which in particular holds for simply connected compact manifolds with strictly convex boundaries such that the geodesic flow has no focal points (lengths of nontrivial Jacobi fields vanishing at a point do not have critical points), in particular if the curvature of the manifold is negative (or just nonpositive). Thus, caustics are still allowed under this condition since a manifold with nonpositive curvature satisfies the condition. Also, as explained in [27], if Ω is a ball and the speeds increase when the distance to the center decreases (typical for geophysical applications), the foliation condition is satisfied.

We denote by u_h the solution to the homogeneous elastic equation on \mathbb{R}^3 , with transmission conditions and initial time Cauchy data h (see (2.2) and (2.3)). All of our function spaces are of the form $X(\cdot; \mathbb{C}^3)$ since we have vector valued functions in the elastic setting, but throughout the paper, we will not write the vector valued part \mathbb{C}^3 to make the notation less burdensome. Let $\bar{\Omega}^c$ be the complement of $\bar{\Omega}$ and we define the *exterior measurement operator* $\mathcal{F} : H_c^1(\bar{\Omega}^c) \oplus L_c^2(\bar{\Omega}^c) \rightarrow C^0(\mathbb{R}_t; H^1(\bar{\Omega}^c)) \cap C^1(\mathbb{R}_t; L^2(\bar{\Omega}^c))$ as (see (2.6) for full details)

$$\mathcal{F} : h_0 \rightarrow u_{h_0}(t)|_{\bar{\Omega}^c}.$$

The operator \mathcal{F} only measures waves outside Ω after undergoing scattering within Ω , and it is associated to a particular elastic operator \mathcal{P} with a set of parameters.¹ See Figure 1 for an example of an initial source h_0 and the measured data $\mathcal{F}h_0$. Given a second set of elastic parameters $\tilde{\lambda}, \tilde{\mu}, \tilde{\rho}$ we obtain analogous operators $\tilde{\mathcal{P}}$ and $\tilde{\mathcal{F}}$. Denote the associated P/S wave speeds $c_{P/S}$ and $\tilde{c}_{P/S}$. The goal of the work is to prove unique determination of μ, λ, ρ from \mathcal{F} under some geometric hypotheses. From here on, we use P/S to refer to either subscript or wave speed. In addition, to avoid the technical difficulties of dealing with corners or higher codimension singularities of $c_{P/S}$, we always assume that the singular support of $c_{P/S}, \tilde{c}_{P/S}$ lies in a closed, not necessarily connected hypersurface in Ω ; we will deal with corners and edges in a separate paper.

We assume the Lamé parameters $\lambda(x)$ and $\mu(x)$ satisfy the *strong convexity condition*, namely that $\mu > 0$ and $3\lambda + 2\mu > 0$ on $\bar{\Omega}$. We also assume that the parameters λ, μ, ρ lie in $L^\infty(\mathbb{R}^3)$ and that λ, μ, ρ are piecewise smooth functions that are singular only on a set of disjoint, closed, connected, smooth hypersurfaces Γ_i of $\bar{\Omega}$, called *interfaces*. We let $\Gamma = \bigcup \Gamma_i$ be the collection of all the interfaces. The two wave speeds are $c_P = \sqrt{(\lambda + 2\mu)/\rho}$ and $c_S = \sqrt{\mu/\rho}$, where ρ is the density. In particular, this ensures that $c_P > c_S$ on $\bar{\Omega}$. As in [7], we will probe Ω with Cauchy data (an *initial pulse*) concentrated close to (but outside $\bar{\Omega}$) Ω with a particular polarization. We will denote by

$$g_P = c_P^{-2} dx^2, \quad g_S = c_S^{-2} dx^2$$

the two different (rough) metrics associated to the rays. As in [7], we can define the distance functions $d_{P/S}(\cdot, \cdot)$ corresponding to the respective metrics by taking the infimum over all lengths of the piecewise smooth paths between a pair of points.

¹We describe the SI units used for measuring various quantities. Acoustic pressure is Pa ($= Nm^{-2}$); the shear modulus μ and the Lamé modulus are usually measured in GPa. The density units are kg/m^3 . The elastic displacements u_{h_0} and $\mathcal{F}h_0$ are measured in m .

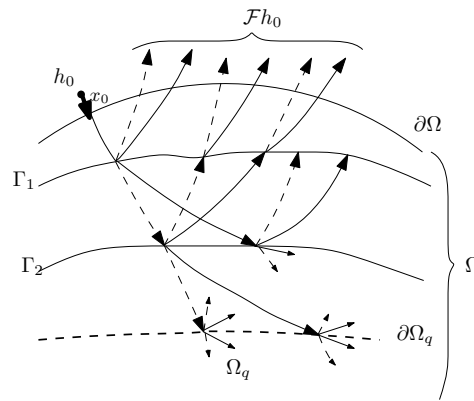


Figure 1. An example of an initial source h_0 (given as time zero Cauchy data) located at x_0 and initially traveling in a given direction. The rest of the diagram depicts the propagation of rays from such an initial source. P waves are denoted with a solid line and S waves are dashed. The measured data is $\mathcal{F}h_0$, which are waves that return and exit Ω . The set Ω_q is an interior subdomain defined in section 3. For simplicity of the diagram, we have suppressed the multiply reflected waves in the measurement data, but they are included in our approach.

Let us define the closed subset $\mathcal{D} := \{x \in \Omega; c_P(x) = 2c_S(x)\} \subset \Omega$. In this article we prove the following result.

Theorem 1.1. Assume $\mathcal{F} = \tilde{\mathcal{F}}$ and that $c_{P/S}, \tilde{c}_{P/S}$ satisfy the extended geometric foliation condition (see section 3). If $(\lambda, \mu, \rho) = (\tilde{\lambda}, \tilde{\mu}, \tilde{\rho})$ in Ω^c , then $(\lambda, \mu, \rho) = (\tilde{\lambda}, \tilde{\mu}, \tilde{\rho})$ inside $\Omega \setminus \mathcal{D}$.

Remark 1.2. We use the map \mathcal{F} for mathematical convenience but we could also consider the time dependent Dirichlet-to-Neumann map on a bounded domain for the elastic wave equation instead. As shown in [21] and [28], the Dirichlet-to-Neumann map determines \mathcal{F} since one can uniquely determine all three material parameters and their derivatives at the boundary from the Dirichlet-to-Neumann map, which then allows one to smoothly extend them outside of Ω to all of \mathbb{R}^3 (they still have the same discontinuities in Ω).

Remark 1.3. We do not need the data to be measured for the whole time \mathbb{R}_t ; instead it is sufficient to take measurement for a finite time $(0, 2T)$ where $T > 0$ is the maximum time that a S -wave takes to travel past Ω . Observe that since $c_P > c_S$, therefore $T > 0$ is enough time for P -waves to travel past Ω . We will give the precise definition of T in section 3.

Remark 1.4. In regard to errors in the parameters, these can be estimated with data in the range of the forward map, in the noise-free case, through stability estimates for the inverse problem. This would require an adaptation analogous to the results in [17] and the references there in the smooth, scalar wave setting. To accommodate noisy data, we will need to extend the current work to a Bayesian framework as developed for the non-abelian geodesic ray transform in [16]. This requires establishing additional quantitative estimates pertaining to injectivity of the outside measurement operator.

Remark 1.5. Note that here we do not consider any correlation between the density and the Lamé parameters. The unknowns λ, μ, ρ are independent piecewise smooth functions satisfying only the strict convexity conditions, i.e., $\mu(x) > 0, \rho(x) > 0$, and $3\lambda(x) + 2\mu(x) > 0$ over $\bar{\Omega}$.

1.1. Outline. In section 2, we introduce the elastic wave equation with transmission conditions. In section 3, we introduce our main geometric assumption which is a convex foliation condition and describe the geometry of bicharacteristics in the elastic setting with interfaces. We also define local travel times and lens relations that we use later in the proof, as well as a standard geometric optics representation of a solution to the elastic wave equation in the presence of scattering. A key ingredient in the main proof is to have a notion of “principal symbol” of an FIO applied to an arbitrary distribution so section 4 is devoted to introducing such a construction based on [31]. The proof of the main theorem is done in section 5, followed by a section with some concluding remarks. The main proof requires past results on recovering wave speeds with discontinuities and recovering all the parameters across a particular interface from reflection data, so those are summarized at the start of section 5.

2. Elastic wave equation and transmission conditions. In this section, let us give the basic definitions and setup of the elastic wave equation that we study in the main theorem. Recall that $\Omega \subset \mathbb{R}^3$ is a smooth, bounded domain. As described in [6, section 2], we will probe Ω with Cauchy data (an initial pulse) concentrated close to Ω with a particular polarization, in some smooth domain $\Theta \supset \Omega$. The wave generated by the source will scatter when it hits an interface inside Ω , and the scattered waves will multiply as they encounter other interfaces (scattered multiples). See [28, Figure 1] for an illustration of the type of domain we consider and the multiple scattering. We will add to the initial pulse a Cauchy data control (a *tail*) supported outside Θ , whose role is to remove multiple reflections up to a certain depth, controlled by a time parameter $T \in (0, \frac{1}{2}\text{diam}_P(\Omega))$, where $\text{diam}_P(\Omega)$ is the diameter of Ω using the metric g_P . See [6, Figure 2] for an illustration of the initial pulse and [6, Figure 3] for the control denoted K_{tail} there. This will require us to consider controls supported in a sufficiently large Lipschitz neighborhood $\Upsilon \subsetneq \mathbb{R}^3$ of Θ . For an incoming elastic wave u_I that enters Ω , we obtain an array of output scattered waves outside Ω after hitting one or more interfaces. From the measurements of the input and the scattered outputs, the goal is to recover the piecewise smooth coefficients in Ω .

Define the P -depth $d_{\Theta}^*(x)$ of a point x inside Θ :

$$d_{\Theta}^*(x) = \begin{cases} +d_P(x, \partial\Theta), & x \in \Theta, \\ -d_P(x, \partial\Theta), & x \notin \Theta. \end{cases}$$

We use the (rough) metric g_P since finite speed of propagation for elastic waves is based on the faster P -wave speed. Since we will probe all of Ω , fix any $T > \text{diam}_S\Omega$. We only require controls supported in a sufficiently large Lipschitz neighborhood $\Upsilon \subsetneq \mathbb{R}^3$ of Θ that satisfies $d_S(\partial\Upsilon, \bar{\Theta}) > 2T$ and is otherwise arbitrary. Thus, we only require elastic wave solutions on the finite time interval $(0, 2T)$.

Following the notation in [28, section 3], for an interface Γ , we work locally in a small neighborhood of a point on Γ and call one of its sides Ω_- negative and the other one Ω_+ , positive. For a material parameter generically denoted c , we have $c = c_-$ in Ω_- and $c = c_+$ in Ω_+ , where c_- , c_+ are smooth up to Γ and $c_- \neq c_+$ pointwise. We denote $f|_{\Gamma_{\pm}}$ to be the limit of $f(x)$ as x approaches Γ from the positive/negative side. Define the Neumann operator at Γ as the normal component of the stress tensor, given as

$$(2.1) \quad \mathcal{N}_{\pm}u = (\lambda \text{div} \otimes \mathbf{I} + 2\mu \hat{\nabla})u \cdot \nu|_{\Gamma_{\pm}},$$

where ν is a fixed unit normal vector at Γ . When we wish to leave the exact boundary restriction ambiguous and just refer to the normal component of the stress tensor at some hypersurface Γ , we will also write $\mathcal{N}u = (\lambda \operatorname{div} \otimes \mathbf{I} + 2\mu \hat{\nabla})u \cdot \nu|_{\Gamma}$ when it is clear which hypersurface Γ we are referring to. The Cauchy problem for the isotropic, inhomogeneous elastic wave equation we consider is

$$(2.2) \quad \begin{aligned} \mathcal{P}u(t, x) &= 0 && \text{in } \mathbb{R}_t \times \mathbb{R}^3, \\ u|_{t=0} = \psi_0 \quad \partial_t u|_{t=0} &= \psi_1, && \text{in } \mathbb{R}^3, \end{aligned}$$

where ψ_0, ψ_1 are compactly supported in $\mathbb{R}^3 \setminus \bar{\Omega}$. We also impose the transmission conditions at each interface Γ_i

$$(2.3) \quad [u] = [\mathcal{N}u] = 0 \quad \text{on } \Gamma_i,$$

where $[v]$ stands for the jump of v from the exterior to the interior across Γ_i . Let us also define the function spaces which will be useful in the analysis. Define the space for the Cauchy data

$$(2.4) \quad \mathbf{C} := H_0^1(\Upsilon; \mathbb{C}^3) \oplus L^2(\Upsilon; \mathbb{C}^3).$$

Next, define F to be the solution operator for the elastic wave initial value problem:

$$(2.5) \quad F : H^1(\mathbb{R}^n) \oplus L^2(\mathbb{R}^3) \rightarrow C(\mathbb{R}, H^1(\mathbb{R}^3)), \quad F(h_0, h_1) = u \text{ s.t. } \begin{cases} \mathcal{P}u = 0, \\ u|_{t=0} = h_0, \\ \partial_t u|_{t=0} = h_1. \end{cases}$$

Thus, the outside measurement map may be written as

$$(2.6) \quad \mathcal{F}(h) = F(h)(t)|_{\bar{\Omega}^c}, \quad h \in \mathbf{C}.$$

Let R_s propagate Cauchy data at time $t = 0$ to Cauchy data at $t = s$:

$$(2.7) \quad R_s = (F, \partial_t F)|_{t=s} : H^1(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3) \rightarrow H^1(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3).$$

3. Geometric assumptions and notation. In this section, we introduce our notation and state the geometric assumptions we make. All the definitions and results in this section are a brief summary of what was defined in [3, 7, 6]. Recall that Γ is the collection of closed connected hypersurfaces Γ_j for $j = 0, 1, \dots, n$ in $\bar{\Omega}$ where $\Gamma_0 = \partial\Omega$ and the parameters are smooth on each connected component of $\mathbb{R}^3 \setminus \Gamma$.

The principal symbol of the hyperbolic operator \mathcal{P} is given as

$$(3.1) \quad p(t, x, \tau, \xi) = -\rho [(\tau^2 - c_S^2 |\xi|^2) I - (c_P^2 - c_S^2) (\xi \otimes \xi)].$$

One can calculate the lower order terms in the full symbol of \mathcal{P} , and the order one term is

$$(3.2) \quad p_1(t, x, \tau, \xi) = -i [\nabla_x \lambda \otimes \xi + (\nabla_x \mu \cdot \xi) I + \xi \otimes \nabla_x \mu].$$

From the principal symbol, we readily observe that ξ and ξ^\perp are eigenvectors of $p(t, x, \tau, \xi)$ with eigenvalues c_P and c_S , respectively. The eigenspace corresponding to c_P is one-dimensional, whereas the eigenspace corresponding to the eigenvalue c_S is two-dimensional.

Geodesics and bicharacteristics. The bicharacteristics curves $\gamma_{P/S}^\pm$ in $T^*(\mathbb{R} \times \mathbb{R}^3)$ are the integral curves of the Hamiltonian vector fields $V_{H_{P/S}^\pm}$, where $H_{P/S}^\pm = \tau \pm c_{P/S}|\xi|$ along with the condition that $\gamma_{P/S}^\pm$ lies in the set $\{\det p(t, x, \tau, \xi) = 0\}$. Parametrized by $s \in \mathbb{R}$, one obtains

$$(3.3) \quad \frac{dt}{ds} = c_{P/S}^{-1}, \quad \frac{dx}{ds} = \pm \frac{\xi}{|\xi|}, \quad \frac{d\tau}{ds} = 0, \quad \frac{d\xi}{ds} = \mp |\xi| \nabla_x (\log c_{P/S}),$$

along with the condition that $\det p(t, x, \tau, \xi) = 0$. We refer to $\gamma_{P/S}^\pm$ as the forward and backward P/S bicharacteristic curves.

Definition 3.1. We refer to a piecewise smooth curve $\gamma_{P/S} : I \subset \mathbb{R} \rightarrow \bar{\Omega}$ as a unit speed broken geodesic in $(\bar{\Omega}, g_{P/S})$ if

1. $\gamma_{P/S}$ is a continuous path which is smooth in $\bar{\Omega} \setminus \Gamma$,
2. each smooth segment of $\gamma_{P/S}$ is a unit speed geodesic with respect to $g_{P/S}$,
3. $\gamma_{P/S}$ intersects Γ at only finitely many points $t_i \in I$ and all the intersections are transversal,
4. $\gamma_{P/S}$ obeys Snell's law of refraction for elastic waves where it cuts Γ .

A broken bicharacteristic is a path in $T^*(\mathbb{R}^n)$ of the form (γ, γ^b) , the flat operation taken with respect to g_P or g_S as appropriate. Note that a broken geodesic defined this way may contain both P and S geodesic segments. More precisely, a broken bicharacteristic (parameterized by a time variable) can be written as $\gamma : (t_0, t_1) \cup (t_1, t_2) \cup \dots \cup (t_{k-1}, t_k) \rightarrow T^*\mathbb{R}^n \setminus \Gamma$, which is a sequence of bicharacteristics connected by reflections and refractions obeying Snell's law: for $i = 1, \dots, k-1$,

$$(3.4) \quad \gamma(t_i^-), \gamma(t_i^+) \in T_\Gamma^*(\mathbb{R}^n), \quad (d\iota_\Gamma)^* \gamma(t_i^-) = (d\iota_\Gamma)^* \gamma(t_i^+),$$

where $\iota_\Gamma : \Gamma \hookrightarrow \mathbb{R}^n$ is the inclusion map, $T_\Gamma^*(\mathbb{R}^n)$ is the cotangent bundle of \mathbb{R}^n that is restricted to Γ in the base variable, d is the differential, and $\gamma(t_i^\mp) = \lim_{t \rightarrow t_i^\mp} \gamma(t)$. Throughout the manuscript, we restrict ourselves to broken geodesics γ that intersect the interfaces transversely (it is never tangent to an interface) since for our geometric optics representation of an elastic wave solution, we assume that solutions have wavefront set disjoint from broken bicharacteristics that are tangent to one of the interfaces. This restriction is justified since as explained in [28, section 5], for the purpose of the inverse problem, it suffices to analyze the propagation of singularities away from a set of measure zero, and thus, there is no need to build a parametrization near the glancing regions. Each restriction $\gamma|_{(t_i, t_{i+1})}$ is a P -bicharacteristic, respectively, S -bicharacteristic, if it is a bicharacteristic for $\partial_t^2 - c_P \Delta$, respectively, $\partial_t^2 - c_S \Delta$. We also refer to each such bicharacteristic as a branch of γ ; we are sometimes more specific and write P branch or S branch if we want to specify the associated metric. For each i , note that $\gamma(t_i)$ projected to the base manifold is a point of Γ_{k_i} for some k_i . A branch $\gamma|_{(t_i, t_{i+1})}$ is reflected if the inner product of $\gamma'(t_i^+)$ and $\gamma'(t_i^-)$ (when projected to base space) with a normal vector to Γ_{k_i} have opposite signs. Otherwise, it is a transmitted branch. Say that $\gamma|_{(t_i, t_{i+1})}$ is a mode converted branch if it is a P/S branch and $\gamma|_{(t_{i-1}, t_i)}$ is a S/P branch.

A purely transmitted P/S broken geodesic (a concatenation of smooth P or S geodesics) is a unit speed broken geodesic that consists of only P/S transmitted branches; that is, the inner products of $\gamma'(t_i^-)$ and $\gamma'(t_i^+)$ with the normal to Γ have identical signs at each t_i and they

are all either P geodesics or S geodesics. A purely transmitted P/S broken bicharacteristic is then defined the same way using projection to base space.

We define the broken P/S bicharacteristics as $(\gamma_{P/S}, \dot{\gamma}_{P/S}^b)$ where γ is a unit speed P/S broken geodesic and the operation (flat) is taken with respect to the metric $g_{P/S}$.

Foliation condition. We assume that the domain Ω has an *extended convex foliation* with respect to both of the metrics g_P and g_S . This is an extension of the convex foliation condition given in [30] to the piecewise smooth setting and was introduced in [6, Definition 3.2].

Definition 3.2 (extended convex foliation). We say $\kappa : \bar{\Omega} \mapsto [0, \tilde{q}]$ is a (piecewise) extended convex foliation for $(\Omega, g_{P/S})$ if

1. κ is smooth and $d\kappa \neq 0$ on $\bar{\Omega} \setminus \Gamma$,
2. κ is upper semicontinuous,
3. each level set $\kappa^{-1}(q)$ is geodesically convex with respect to g_P and g_S , when viewed from $\kappa^{-1}((q, \tilde{q}))$ for any $t \in [0, \tilde{q}]$,
4. $\partial\Omega = \kappa^{-1}(0)$ and $\kappa^{-1}(\tilde{q})$ has measure zero,
5. there is some $q_i \in [0, \tilde{q}]$ such that $\Gamma_i \subset \kappa^{-1}(q_i)$ for $i = 0, \dots, m$,
6. $\limsup_{\epsilon \rightarrow 0^+} c_{P/S}|_{\kappa^{-1}(q+\epsilon)} \leq \limsup_{\epsilon \rightarrow 0^+} c_{P/S}|_{\kappa^{-1}(q-\epsilon)}$ whenever $\Gamma_i \subset \kappa^{-1}(q)$ for some i and Γ_i .

We say (c_P, c_S) satisfies the extended foliation condition if there exists an extended convex foliation for $(\Omega, c_{P/S})$.

See the discussion below [6, Definition 3.2] for an explanation of the last condition. We write $\Omega_q := \kappa^{-1}(q, \tilde{q}]$ to denote the part of the domain whose boundary is $\Sigma_q := \kappa^{-1}(q)$. Let us fix the convention of writing “above” Σ_q to be outside of Ω_q and “below” to be inside Ω_q . We write Σ_q^\mp to denote two copies of Σ_q approached from “above” or “below” Σ_q . Let us observe that, if required, we can extend each Γ_j along with Σ_{q_j} and denote Ω_j to be the connected components of $\bar{\Omega} \setminus \Gamma$. Write $\tilde{\Omega}_j$ to be Ω_{q_j} , where $\Gamma_j \subset \Sigma_{q_j}$.

Definition 3.3. We define the set of the inward or the outward pointing covectors at a closed connected hypersurface Σ as

$$T_{\pm}^*\Sigma := \{(x, \xi) \in T^*\Sigma : \pm \langle \xi, d\kappa \rangle > 0\},$$

where the above inner product is taken in the Euclidean sense.

For a domain Ω_q we define

$$\partial_{\pm} T^*\Omega_q := \{(x, \xi) \in T^*\Omega; \pm \langle \xi, d\kappa \rangle > 0\}.$$

With the help of the notion of the inward or outward covectors, we define the q -interior travel time and lens relation in Ω .

Definition 3.4. Let $(x, \xi) \in T_{+}^*\Sigma_q \setminus \{0\}$. Let $\gamma_{P/S}$ be the unit speed broken geodesic such that

$$\lim_{t \rightarrow 0^+} (\gamma_{P/S}(t), \dot{\gamma}_{P/S}(t)) = (x, \xi).$$

- (i) We define the q -interior travel time $l_{P/S,q}(x, \xi) > 0$ such that $\gamma_{P/S}(l_{P/S,q}) \in \Sigma_q$.

- (ii) The q -interior P/S lens relation is $L_{P/S,q}(x, \xi) := (\gamma_{P/S}(l_{P/S,q}), \dot{\gamma}_{P/S}(l_{P/S,q})) \in T_-^* \Sigma_q$, where $l_{P/S,q}(x, \xi)$ is defined as above.

We end this section by summarizing all the assumptions we have made so far.

Assumption 1. Let us assume all the notations and definitions above. We assume that

- (i) the interfaces Γ_j are a collection of disjoint, connected, closed hypersurfaces in $\bar{\Omega}$,
(ii) Ω has an extended convex foliation κ with respect to both the metrics $g_{P/S}$.

FIOs and elastic parametrix. In this section we show the microlocal parametrix construction for the system (2.2) in the smooth setting similar to [20]. Since many of our arguments in the proof of the main theorem are local, this will suffice. The full parametrix with transmission conditions where the material parameters have discontinuities is in Appendix B. First, the geometric optics for the initial value problem (2.2) when the material parameters are smooth has the form

$$U = E_0 f_0 + E_1 f_1,$$

where E_k , $k = 0, 1$, are the solution operators given in terms of FIOs. We write $A \equiv B$ for two FIOs A and B to denote that A is same as B modulo a smoothing operator. We impose that the FIOs E_0 and E_1 solve the following system modulo a smoothing operator:

$$(3.5) \quad \begin{aligned} \mathcal{P}E_k &\equiv 0 && \text{on } (0, \infty) \times \mathbb{R}^3, \\ E_0|_{t=0} &\equiv I, \quad \partial_t E_0|_{t=0} \equiv 0, && \text{on } \mathbb{R}^3, \\ E_1|_{t=0} &\equiv 0, \quad \partial_t E_1|_{t=0} \equiv I, && \text{on } \mathbb{R}^3. \end{aligned}$$

The FIOs E_k for $k = 0, 1$ are given as

$$(3.6) \quad E_k v = \sum_{\pm, P/S, l} \int_{\mathbb{R}^3} e^{i\phi_{P/S}^\pm(t, x, \xi)} a_{\pm, k, P/S}^{\cdot, l}(t, x, \xi) \hat{v}_l(\xi) d\xi, \quad \text{where } v = (v_1, v_1, v_3),$$

$a_{\pm, k, P/S}^{\cdot, l}(t, x, \xi)$ is a matrix valued symbol, \hat{v} is the Fourier transform of v (componentwise), and $i = \sqrt{-1}$. The phase function $\phi_{P/S}^\pm(t, x, \xi)$ is homogeneous of order 1 in ξ and solves the eikonal equation

$$(3.7) \quad \det p(t, x, \partial_t \phi_{P/S}^\pm, \nabla_x \phi_{P/S}^\pm) = 0.$$

One can simplify the eikonal equation to

$$(3.8) \quad \partial_t \phi_{P/S}^\pm = \mp c_{P/S} |\nabla_x \phi_{P/S}^\pm|.$$

We can choose the initial value to be $\phi_{P/S}^\pm|_{t=0} = x \cdot \xi$ and solve the above equation using Hamilton–Jacobi theory.

Remark 3.5. Note that the phase function $\phi_{P/S}^\pm$ can be determined by the principal symbol p and thus the wave speeds $c_{P/S}$. The principal symbol of E_k is also determined by the wave speeds and not the density [20]. Therefore, if one needs to recover the parameters ρ, λ, μ individually, then one must consider the lower order terms of the asymptotic expansion of $a_{\pm, k, P/S}^{\cdot, l}$.

Unlike the phase function $\phi_{P/S}^\pm$, the amplitudes cannot be determined using only the principal symbol p . We do an asymptotic expansion of the amplitudes $a_{\pm,k,P/S}^{j,l}(t, x, \xi)$ as

$$a_{\pm,k,P/S}^{j,l}(t, x, \xi) = \sum_{J=0,-1,-2,\dots} \left(a_{\pm,k,P/S}^{j,l} \right)_J,$$

where $(a_{\pm,k,P/S}^{j,l})_J$ are homogeneous of order J in $|\xi|$. Now, each 3×3 matrix $(a_{\pm,k,P/S}^{j,l})_J(t, x, \xi)$ satisfies

$$(3.9) \quad p(t, x, \partial_t \phi_{P/S}^\pm, \nabla_x \phi_{P/S}^\pm) (a_{\pm,k,P/S})_{J-1} = B_{P/S} (a_{\pm,k,P/S})_J + C_{P/S} (a_{\pm,k,P/S})_{J+1},$$

where $(a_{\pm,k,P/S})_1 = 0$ and the matrix operators $B_{P/S}, C_{P/S}$ are given as

$$(3.10) \quad \begin{aligned} (B_{P/S}V) &= i\partial_{\tau,\xi} p \left(t, x, \partial_t \phi_{P/S}^\pm, \nabla_x \phi_{P/S}^\pm \right) \cdot \partial_{t,x} V + ip_1(t, x, \partial_t \phi_{P/S}^\pm, \nabla_x \phi_{P/S}^\pm) V \\ &\quad + i\frac{1}{2} \sum_{|\beta|=2} \partial_{\tau,\xi}^\beta p \left(t, x, \partial_t \phi_{P/S}^\pm, \nabla_x \phi_{P/S}^\pm \right) \cdot \left(\partial_{t,x}^\beta \phi_{P/S}^\pm \right) V, \\ (C_{P/S}V) &= i\partial_{\tau,\xi} p_1 \left(t, x, \partial_t \phi_{P/S}^\pm, \nabla_x \phi_{P/S}^\pm \right) \cdot \partial_{t,x} V \\ &\quad + \frac{1}{2} \sum_{|\beta|=2} \partial_{\tau,\xi}^\beta p \left(t, x, \partial_t \phi_{P/S}^\pm, \nabla_x \phi_{P/S}^\pm \right) \cdot \partial_{t,x}^\beta V. \end{aligned}$$

In order to calculate the explicit form of $(a_{\pm,k,P/S}^{j,l})_J$, let us define the unit vectors $N = \frac{\nabla_x \phi_P^\pm}{|\nabla_x \phi_P^\pm|}$ and define N_1, N_2 , such that $\{N_1, N_2\}$ forms an orthonormal basis of the kernel of $p(t, x, \partial_t \phi_S^\pm, \nabla_x \phi_S^\pm)$. Observe that the unit vector N spans the kernel of $p(t, x, \partial_t \phi_P^\pm, \nabla_x \phi_P^\pm)$ and $\{N_1, N_2\}$ form an orthonormal basis for the kernel of $p(t, x, \partial_t \phi_S^\pm, \nabla_x \phi_S^\pm)$. Now, let us write

$$(3.11) \quad \begin{aligned} (a_{\pm,k,P}^{j,l})_J &= (h_{\pm,k,P}^{j,l})_J + (b_{\pm,k,P}^{j,l})_J N, \quad l = 1, 2, 3, \\ (a_{\pm,k,S}^{j,l})_J(t, x, \xi) &= (h_{\pm,k,S}^{j,l})_J + \left[(b_{\pm,k,S}^{j,l})_{1,J} N_1 + (b_{\pm,k,S}^{j,l})_{2,J} N_2 \right], \end{aligned}$$

where $(h_{\pm,k,P/S}^{j,l})_J$ is perpendicular to the kernel of $p(t, x, \partial_t \phi_{P/S}^\pm, \nabla_x \phi_{P/S}^\pm)$ for $J \leq -1$ with $(h_{\pm,k,P/S}^{j,l})_0 = 0$.

Observe that from (3.9) we obtain a necessary condition that

$$(3.12) \quad N_{P/S} \left[B_{P/S} (a_{\pm,k,P/S})_J + C_{P/S} (a_{\pm,k,P/S})_{J+1} \right] = 0 \quad \text{for } J = 0, -1, -2, \dots,$$

where $N_P = N$ and $N_S = N_1, N_2$.

4. Symbol of an FIO applied to an arbitrary distribution. In a layer stripping procedure to recover the density of mass, we will use scattering control to generate internal sources with a specified wavefront set (as done in [28, 6]). The data that we will be able to recover from the outside measurement operator has the form FV where V is more or less an arbitrary

distribution (the internal source we want to generate) and F is an FIO representation of the elastic wave propagator. We need to recover the terms in the polyhomogeneous expansion of the symbol of F from such data and in particular show that they vanish when doing a uniqueness argument. The main idea is quite natural and an analogous argument can be found in [11, Example 2.6]. When A is a PsiDO and u is a compactly supported distribution that is smooth outside $\{0\}$, $\widehat{Au}(\xi)$ has an expansion involving the full symbol of A and its derivatives, together with \widehat{u} and its derivatives, evaluated at $(0, \xi)$. When $Au \in C^\infty$, then the terms in this expansion vanish, which creates algebraic equations where one can extract information regarding the principal symbol of A as well as its lower order symbols in the classical symbol expansion. Our aim is to do something similar when A is an FIO whose associated Lagrangian is a canonical graph, and the formalism of Weinstein's symbol calculus in [31] will be natural to obtain a similar expansion by associating a principal symbol to an arbitrary distribution.

In a seminal paper [31], Weinstein showed how to define the principal symbol of an arbitrary distribution and showed a product type formula for the principal symbol of a PsiDO applied to an arbitrary distribution. Here, we tweak several arguments in [31] to compute the principal symbol of a canonical graph FIO applied to a distribution. We state the necessary lemmas and propositions we will use in this manuscript but postpone the proofs to Appendix A to not interrupt the flow of the article.

4.1. Weinstein symbols of arbitrary distributions. We use the formulation of Weinstein to define the symbol of a distribution. First, fix $(x_0, \xi) \in T^*\mathbb{R}^n$. Let $f \in \mathcal{D}'(\mathbb{R}^n)$ and let $\phi : \mathbb{R}^n \rightarrow \mathbb{R}$ be a C^∞ function with $\phi(x_0) = 0$ and $d\phi(x_0) = \xi$. We define the distribution f_ϕ^τ for $\tau \geq 1$ by

$$\langle f_\phi^\tau, u \rangle = \langle f, \tau^{n/2} e^{-i\tau\phi(x)} u(\sqrt{\tau}(x - x_0)) \rangle,$$

where u is a test function. If $f \in L^1$, then $f_\phi^\tau = (f e^{-i\tau\phi})(x - x_0)/\sqrt{\tau}$. Essentially, $f_\phi^\tau = T_{x_0} D_{1/\sqrt{\tau}} M_{e^{-i\tau\phi}} f$, where M_\bullet is multiplication operator, D_\bullet is dilation operator, and T_{x_0} is a translation operator so this distribution is well defined. We will usually use $\phi = (x - x_0) \cdot \xi$ so we leave out the subscript ϕ in this case and just write g^τ . It is also useful to denote $u_\tau = \tau^{n/2} e^{-i\tau\phi(x)} u(\sqrt{\tau}(x - x_0))$ so that $\langle f^\tau, u \rangle = \langle f, u^\tau \rangle$.

We also need a notion of "order" of a distribution by measuring the growth rate of f^τ . We have the following definition from [31, Definitions 1.1.3, 1.1.7, 1.3.6].

Definition 4.1. *If \mathfrak{E} is a vector space with a distinguished class of subsets called "bounded sets," we denote by $S^N(\mathfrak{E})$ the set of families $[1, \infty) \ni \tau \mapsto f^\tau \in \mathfrak{E}$ for which the set $\{\tau^{-N} f^\tau | \tau \geq 1\}$ is bounded. If $\tau \mapsto f^\tau$ belongs to $S^N(\mathfrak{E})$, we write $f^\tau = O(\tau^N)$. If the set $S = \{N | f^\tau = O(\tau^N)\}$ is of the form $[a, \infty)$, we define the order of f at (x_0, ξ) to be a and denote it by $O_{x_0, \xi_0}(f)$. If $S = (-\infty, \infty)$, we define $O_{x_0}(f) = -\infty$.*

Let $f \in \mathcal{D}'(\mathbb{R}^n)$ be such that $O_{(0, \xi)}(f) \leq N$. We define the Weinstein symbol of order N

$$\sigma_{(x_0, \xi)}^N(f)$$

of f at $(x_0, \xi) \in T^\mathbb{R}^n$ as the image in $S^N(\mathcal{D}'(\mathbb{R}^n))/S^{N-1}(\mathcal{D}'(\mathbb{R}^n))$ of the family f_ϕ^τ under the natural map $S^N(\mathcal{D}'(\mathbb{R}^n)) \rightarrow S^N(\mathcal{D}'(\mathbb{R}^n))/S^{N-1}(\mathcal{D}'(\mathbb{R}^n))$.*

This is a slight deviation from [31, Definition 1.3.6] which instead involves ϕ and certain equivalence classes, but as mentioned there, in computing $\sigma_{(x_0, \xi)}^N(f)$ for a specific f , it suffices to determine the behavior modulo $S^{N-1}(\mathcal{D}'(\mathbb{R}^n))$ of the family f_ϕ^τ for a particular ϕ with $d\phi(x_0) = \xi$. In this paper, we will only use $\phi(x) = (x - x_0) \cdot \xi$.

Also, we have the following definition from [31, Definition 1.3.12].

Definition 4.2. *A distribution f is homogeneous of degree N at (x_0, ξ) if for a particular ϕ with $d\phi(x_0) = \xi$, $f_\phi^\tau = \tau^N \gamma + O(\tau^{N-1/2})$, where γ is a fixed element of $\mathcal{D}'(\mathbb{R}^n)$. This is equivalent to the definition in [31] and then the symbol of f can be identified with γ as described there.*

For the distributions we consider here, it will suffice to pair f^τ with an arbitrary test function and determine the leading order term in τ as $\tau \rightarrow \infty$.

As described in [31, section 1.6], the above definitions and concepts naturally extend to vector valued distributions on a manifold using natural identifications

$$\mathcal{D}'(\mathbb{R}^n, V) \approx \mathcal{D}'(\mathbb{R}^n) \otimes_{\mathbb{C}} V,$$

where V is a finite dimensional vector space over \mathbb{C} .

4.2. Symbol of a PsiDO applied to a distribution. Here we state a few results which are minor modifications of results in [31], whose proofs we provide in Appendix A. Recall that Ψ^m is the set of pseudodifferential operators of order m .

Lemma 4.3. *Let $P \in \Psi^m$ be properly supported with a principal symbol representative $p_m(x, \xi)$. Let $f \in \mathcal{D}'(\mathbb{R}^n)$ and $(x_0, \xi) \in T^*\mathbb{R}^n \setminus 0$ with $O_{(x_0, \xi)}(f) \leq N$. Then $O(Pf) \leq m + N$ and*

$$\sigma_{(x_0, \xi)}^{N+m}(Pf) = p_m(x_0, \tau\xi)\sigma_{(x_0, \xi)}^N(f).$$

In fact, given a test function $u \in \mathcal{D}(\mathbb{R}^n)$, we have

$$Pu_\tau(x) = p_m(x_0, -\tau\xi)u_\tau(x) + O(\tau^{m-1/2})$$

and

$$\langle (Pf)^\tau, u \rangle = p_m(x_0, \tau\xi)\langle f^\tau, u \rangle + O(\tau^{m+N-1/2})$$

as $\tau \rightarrow \infty$.

We also need to know how a diffeomorphism transforms the symbol. First, using the proof of [31, Proposition 1.4.1], we have the following.

Proposition 4.4. *Let $\theta : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a diffeomorphism such that $\theta(x_0) = y_0$. Then $O_{y_0, \phi}(f) \leq N$ implies*

$$(f \circ \theta)_{\phi \circ \theta}^\tau - f_\phi^\tau \circ d_{x_0}\theta = O(\tau^{N-1/2}),$$

where $d_{x_0}\theta : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is the derivative of θ at x_0 .

It will be useful to write out each term in the above proposition. Explicitly, we have

$$(4.1) \quad \tau^{-N/2} \langle (f \circ \theta)_{\phi \circ \theta}^\tau, u \rangle = \langle f \circ \theta, e^{i\tau\phi(\theta(x))} u(\sqrt{\tau}(x - x_0)) \rangle = \langle f, J_{\theta^{-1}}(y) e^{i\tau\phi(y)} u(\sqrt{\tau}(\theta^{-1}(y) - x_0)) \rangle,$$

where J_h denotes the Jacobian of h . Then modulo $O(\tau^{N-1/2})$, this is equal to

$$(4.2) \quad \begin{aligned} \tau^{-n/2} \langle f_\phi^\tau \circ d_{x_0} \theta, u \rangle &= \tau^{-n/2} \langle f_\phi^\tau, J_{\theta^{-1}}(y_0) u((d_{x_0} \theta)^{-1} y) \rangle \\ &= \langle f, e^{-i\tau\phi(y)} J_{\theta^{-1}}(y_0) u(\sqrt{\tau}(d_{x_0} \theta)^{-1}(y - y_0)) \rangle. \end{aligned}$$

For the principal symbol, observe that

$$\phi \circ \theta(x_0) = 0, \quad d_{x_0}(\phi(\theta)) = (\theta'_x(x_0))^t \phi'(y_0) = (\theta'_x(x_0))^t \eta_0 = \theta^* \eta_0.$$

Thus, $\phi \circ \theta$ is a phase function for (x_0, ξ_0) where $\xi_0 = \theta^* \eta_0$. Notice θ induces a map on $\mathcal{D}'(\mathbb{R}^n)$ by $h \mapsto h \circ d_{x_0} \theta$ which preserves bounded sets. It also induces a map on the phase functions via $[\phi] \mapsto [\phi \circ \theta]$ that induces a map on principal symbols.

Corollary 4.5. *With the setup as above, we have*

$$\sigma_{x_0, \theta^* \eta_0}(f \circ \theta) = \sigma_{y_0, \eta_0}(\theta)[\sigma_{y_0, \eta_0}(f)].$$

4.3. Principal symbol of an FIO applied to a distribution. Let us consider an FIO $A \in I^\mu(Y \times X, \Lambda_\chi)$, which is the space of FIOs associated to a Lagrangian as defined in [15, Chapter 25], where X and Y are two manifolds and the associated Lagrangian Λ_χ is a canonical graph of a symplectomorphism $\chi : T^*X \rightarrow T^*Y$. In fact, the only case we need is for $X = \mathbb{R}^n$ and $Y = \mathbb{R}^n$, so let us assume this to simplify the notation. Our proofs apply to operators acting on vector bundles as well, and with Hörmander's notation [15, Chapter 25], this includes operators in the FIO space on vector bundles denoted $I^\mu(Y \times X, \Lambda_\chi; \text{Hom}(E, F))$ for vector bundles E and F . We can write

$$\Lambda = \Lambda_\chi = \{(\chi(x, \xi)), (x, \xi)\} \subset T^*Y \times T^*X.$$

The order of A is μ and can be written in the form

$$A = \int e^{i(S(y, \xi) - x \cdot \xi)} a(y, \xi) d\xi$$

and $a \in S^\mu$. There is the associated set

$$C_\Phi = \{(y, x, \xi) : d_\xi \Phi = 0\}$$

and a diffeomorphism $T_\Phi : C_\Phi \rightarrow \Lambda$. Via a projection, we can view a as a symbol over C_Φ and hence Λ via the diffeomorphism T_Φ . We can write explicitly

$$\Lambda = \{y, S'_y, S'_\xi, \xi\}$$

as the graph of the canonical transformation

$$\chi : (S'_\xi, \xi) \mapsto (y, S'_y).$$

Following the notation and terminology in [1, section 2], for fixed ξ_0 , we introduce the *coordinate transform*

$$y \mapsto S'_\xi(y, \xi_0) := T(y) = x(y),$$

which is a local diffeomorphism. Denote the Jacobian determinant $J_T(y) := |\partial T/\partial y|$. In the proof below, we will be using the adjoint A^t , but the adjoint is also associated to a canonical graph and may be put in the form above, which we shall use.

We aim to prove the following proposition which lets us compute a simple form for the Weinstein principal symbol of an FIO associated to a canonical graph applied to a distribution.

Proposition 4.6. *Let $(x_0, \xi_0) = \chi(y_0, \eta_0)$ and $f \in \mathcal{D}'(Y)$. Let $A \in \mathcal{I}^\mu(X \times Y; \Lambda_\chi)$ so that the distributional adjoint $A^t \in \mathcal{I}^\mu(Y \times X; \Lambda_{\chi^{-1}})$ has a representation of the form $A^t = \int e^{i(S(y, \xi) - x \cdot \xi)} a^t(y, \xi) d\xi$ where a^t has a polyhomogeneous expansion with principal term denoted a_μ^t , homogeneous of degree μ in the ξ variable. Then $y_0 = T^{-1}(x_0)$ and*

$$\sigma_{x_0, \xi_0}(Af) = a_\mu^t(y_0, -\tau\xi_0) J_{T^{-1}}(x_0) \sigma_{x_0, \xi_0}(f \circ T^{-1}).$$

In this paper, we are interested in $A \in \mathcal{I}^\mu(X \times Y, \Lambda_\chi)$ with a representation

$$A = \int e^{i(\phi(x, \eta) - y \cdot \eta)} a(x, \eta) d\eta,$$

where $a = a_{(\mu)} + \tilde{a}$ with $\tilde{a} \in S^{\mu-1}$ and $a_{(\mu)}$ homogeneous in η of degree μ . Following Hörmander in [15, Chapter 25], denoting $s : Y \times X \rightarrow X \times Y$ as a function that interchanges the two factors, then $s^* a_{prin}^t$ is the principal symbol of A^* if a_{prin} is the principal symbol of A and a_{prin}^t is the matrix transpose when the vector bundles have been trivialized (in our setting; both vector bundles are \mathbb{C}^3 so there is no need to specify a trivialization). As shown in [15, Chapter 25], the principal symbol of A is well defined and determined by a $1/2$ -density over Λ and a Maslov bundle factor that are both determined by Λ . Hence, a_μ^t computed in the above proposition and $s^* a_{prin}^t$ restricted to Λ can only differ by a factor, denoted J_Λ , which is completely determined by the Lagrangian Λ (since Λ is a canonical graph, there is a natural trivialization of a_{prin} described in [15, Chapter 25]). If we denote the polyhomogeneous expansion as $a \sim \sum_j a_{(\mu-j)}$, then the preceding discussion implies

$$a_\mu^t|_\Lambda = J_\Lambda s^* a_{(\mu)}^t|_\Lambda.$$

We then get the following important corollary using the definition of the Weinstein symbol.

Corollary 4.7. *With the notation and assumptions in Proposition 4.6, and f a distribution of order N , we have*

$$(4.3) \quad \liminf_{\tau \rightarrow \infty} \tau^{-\mu-N} \langle Af, u_\tau \rangle = a_{(\mu)}^t(y_0, \xi_0) J_\Lambda(y_0) \tau^{-N} \liminf_{\tau \rightarrow \infty} \langle f \circ T^{-1}, u_\tau \rangle < \infty,$$

where J_Λ is a quantity determined by Λ .

5. Recovery of the parameters. In this section, we will list the preliminary results and ingredients needed to prove Theorem 1.1 and then give the final proof in section 5.2. Since this is an intricate procedure with various pieces, we provide a summary of the proof. Note that, with a suitable choice of foliation, we can identify $\bar{\Omega}$ with the disjoint union

$$\bar{\Omega} = \bigsqcup_{j=0}^m \bar{\Omega}_j, \quad \text{where } \Gamma_j \subseteq \Sigma_{q_j} = \kappa^{-1}\{q_j\},$$

and $\Omega_j = \kappa^{-1}([q_j, q_{j+1}])$ are defined via Definition 3.2. Here we have $0 = q_0 < q_1 < \dots < q_m = q_{m+1}$ and $|\Omega_m| = 0$.

Sketch of the proof of Theorem 1.1. We prove the result using a layer stripping argument. We split the proof into two main parts.

In the first part, we fix a $q \geq 0$ and assume that the elastic parameters are known on $\Omega_q^* := \Upsilon \setminus \overline{\Omega}_q$. Using the microlocal parametrix in the previous section, we construct Cauchy data h on Ω^* such that in time $T > 0$, the wave field U_h reaches Ω_q and we can specify the wavefront set as well as the mode of the singularity (i.e., whether the singularity is on P or S bicharacteristics) of the solution at Σ_q . We then add a suitable control (which is semiexplicit) to the Cauchy data, so that when propagated, it cancels the multiple scattering of the wave field after time $T > 0$ so that we essentially generate a virtual elastic source at an interior point in Ω , with an initial wavefront set given by a chosen codirection and a desired polarization.

With such specialized waves, we can recover q -interior travel times and q -interior lens data on Σ_q^+ . Using the local boundary rigidity result of Stefanov, Uhlmann, and Vasy [25], we determine the wave speeds $c_{P/S}$ on a neighborhood \mathcal{O} of Σ_q^+ in $\overline{\Omega}_q$. Here we use the fact that Ω has a strong convex foliation aligned with the interfaces. This much was already proven in [6] and [28] so it remains to recover information on the density of mass. If Σ_q contains an interface, we measure the reflected amplitudes of such carefully constructed waves to determine the reflection operator M_R on Σ_q^- . Having the knowledge of M_R on the boundary, we use the result of [4, Theorem 1.1] to recover λ, μ, ρ and all their derivatives at Σ_q^+ . With the specialized controls in the Cauchy data, we generate P -waves that are singular along a P -ray inside Ω_q that connects two nearby points on Σ_q^+ . We can then use the Weinstein symbols discussed in section 4 to recover the lower order amplitudes of such waves. These are determined by $a_{\pm, k, P/S}$ restricted to a corresponding Lagrangian that we described earlier.

In the second part, we consider an asymptotic expansion of the amplitude function $a_{\pm, k, P/S}$ on the P bicharacteristics in $T^*\Omega_q$, starting and ending at $T^*\Sigma_q$. We observe that along the P bicharacteristics, the terms in the asymptotic expansion of the amplitude satisfy transport equations with the initial data prescribed at Σ_q . Here, the boundary data consists of the solution and its Neumann derivative at the boundary Σ_q of Ω_q .

By varying the boundary data, we will recover the local geodesic ray transform of a 2-tensor A_ρ given by the double derivatives of the density function ρ in $\mathcal{O} \subset \overline{\Omega}_q$. Using the injectivity result [26] on the geodesic ray transform of 2-tensors, we recover the action of the *Saint-Venant operator* on A_ρ . In other words, we determine a fourth order elliptic PDE that ρ satisfies in $\mathcal{O} \subset \overline{\Omega}_q$. Using elliptic unique continuation results, we recover the density function ρ in the neighborhood \mathcal{O} outside of the set \mathcal{D} , where the PDE is not elliptic. Having $c_{P/S}$ and ρ , we obtain the parameters λ, μ in \mathcal{O} . We proceed by iteration and finally recover the parameters everywhere in Ω . In the next subsection, we list the key ingredients and previous results that will be essential to the main proof.

5.1. Summary of preliminary results.

5.1.1. Local travel time tomography. A key ingredient in the proof of uniqueness will be the following theorem proved by Stefanov, Uhlmann, and Vasy in [25].

Theorem 5.1. *Choose a fixed metric g_0 on Ω . Let $n = \dim(\Omega) \geq 3$; let $c, \tilde{c} > 0$ be smooth, and suppose $\partial\Omega$ is convex with respect to both $g = c^{-2}g_0$ and $\tilde{g} = \tilde{c}^{-2}g_0$ near a fixed $x \in \partial\Omega$. If $d_g(x_1, x_2) = d_{\tilde{g}}(x_1, x_2)$ for x_1, x_2 on $\partial\Omega$ near x , then $c = \tilde{c}$ in Ω near x .*

We write down a trivial corollary due to continuity of the distance function.

Corollary 5.2. *Consider the same setup as in the above theorem. If $d_g(x_1, x_2) = d_{\tilde{g}}(x_1, x_2)$ for a dense set of points x_1, x_2 on some neighborhood of x in $\partial\Omega$, then $c = \tilde{c}$ in Ω near x .*

We need this since due to the multiple scattering in our setting, we will only be able to recover boundary travel times on a dense set of points and not a full neighborhood. We quote a similar result for the lens relation. Let L denote the lens map.

Corollary 5.3 (see [25, Corollary]). *Let Ω, c, \tilde{c} be as above with $c = \tilde{c}$ on $\partial\Omega$ near x . Let $L = \tilde{L}$ near $S_x\partial\Omega$, which is the set of unit tangent vectors to $\partial\Omega$ at x . Then $c = \tilde{c}$ in Ω near x .*

As before, the same corollary holds if we instead assume $L = \tilde{L}$ in a set that is dense inside some neighborhood of $S_p\partial\Omega$.

5.1.2. Recovery of wave speeds and density of mass across an interface from reflected waves. In this section, we show that if all material parameters have already been recovered from \mathcal{F} within a layer up to an interface Γ_i , then the parameters as well as their normal derivatives can be recovered infinitesimally across Γ_i ; that is, $\partial_\nu^J \rho^{(+)}, \partial_\nu^J \lambda^{(+)}, \partial_\nu^J \mu^{(+)}$ can be recovered as well (see Corollary 5.8). We will do this by analyzing the amplitudes of waves reflected at Γ_i from above. Let us define the incoming and outgoing unit sphere bundle $\partial_\pm S^* \Omega_q := \{(x, \xi) \in \partial_\pm T^* \Omega_q; |\xi| = 1\}$. We start with a key proposition.

Proposition 5.4. *Let us fix $(x, \xi) \in \partial_+ S^* \Omega_q$. Let V be a distribution, such that $WF(V) = \{(x, s\xi); s \in \mathbb{R} \setminus \{0\}\}$, supported outside Ω_q for some $q > 0$. There exist a large enough $T > 0$ and a Cauchy data $U_\infty \in \mathbf{C}$ supported in Ω^* such that*

$$(5.1) \quad WF(\mathcal{F}_{T+s} U_\infty - \mathcal{F}_s V) = \emptyset, \quad \text{for } s \geq 0, \quad \text{in } \Omega_q.$$

Moreover, one can arrange that the singularity of U_∞ flows along the P characteristics γ_P^\pm outside Σ_q , i.e., $WF(U_\infty) \subset \gamma_P^\pm$ for time t close enough to T . Similarly one may take $WF(U_\infty) \subset \gamma_S^\pm$ for time t close enough to T .

Proof. We will give a brief sketch of the proof, since it is similar to [6, Proposition 5.3]. First we prove the result for $(x, \xi) \in \mathcal{S} \subset T^* \Omega$ consists of (x, ξ) such that all the *bad bicharacteristics*² through (x, ξ) are *(+)-escapable*. We consider a P/S purely transmitted bicharacteristic $\gamma_{P/S}$ through (x, ξ) outside Ω_τ such that $\gamma_{P/S}(T) = (x, \xi)$ and $\gamma_{P/S}(0) \in \partial\Theta$. Consider a Cauchy data U_0 supported in $\Theta \setminus \bar{\Omega}$ such that $WF(U_0) = \{(\gamma(0), s\dot{\gamma}(0)); s \in \mathbb{R}_+\}$. Using finite speed of propagation and the fact that the singularity flows along the bicharacteristics, one obtains $WF(\mathcal{F}_T U_0 - V) = \emptyset$. As in [6, Appendix C], one can construct a tail denoted K_{tail} , that together with U_0 cancels the multiple scattering and $U_\infty := U_0 + K_{tail}$ is the required Cauchy data. That construction relies on M_T to be elliptic away from the glancing set on

²See [6, Definition C.1].

both sides of an interface, and this is independent of ρ , so the same proof holds. Finally, using a density argument (as in [6, Lemma 5.9]), we remove the restriction $(x, \xi) \in \mathcal{S}$. ■

Remark 5.5. The usefulness of the above proposition is in the layer stripping argument, where we recover the parameters outside Ω_τ and probe the Cauchy data U_∞ outside Ω , which generates singularities along the bicharacteristics in Ω_τ . Furthermore, one can generate the flow of singularity along the P and the S bicharacteristics separately.

Remark 5.6. The time $T > 0$ is determined by the time it takes for all the branches of the scattered waves to travel from Ω^* to Σ_q and back to Θ^* . Essentially, T can be estimated by the S distance between Ω^* and $x \in \Sigma_q$. For details see [6, Remark 5.4].

We now have an important lemma, taken from [6, Lemma 5.6], that allows us to recover the reflection operator from reflected waves measured outside Ω .

Lemma 5.7. *Suppose that $\Sigma_q \subset \Gamma$ and $c_{P/S} = \tilde{c}_{P/S}, \rho = \tilde{\rho}$ outside $\bar{\Omega}_q$. Assume $\mathcal{F} = \tilde{\mathcal{F}}$. Then*

$$M_R \equiv \tilde{M}_R \text{ on } T^*\Sigma_q^-.$$

Proof. The proof is identical to that of [6, Lemma 5.6]. In that paper, the assumption that $\rho = 1$ was only needed to have $P = \tilde{P}$ outside $\bar{\Omega}_q$, which is also true in our case due to the assumptions given. ■

We then have two important corollaries that follow from Theorem 1.5 in [4].

Corollary 5.8. *Suppose that $\Sigma_q \subset \Gamma$ and $c_{P/S} = \tilde{c}_{P/S}, \rho = \tilde{\rho}$ outside $\bar{\Omega}_q$. Assume $\mathcal{F} = \tilde{\mathcal{F}}$. Then $\partial_\nu^j c_{P/S}^{(+)} = \partial_\nu^j \tilde{c}_{P/S}^{(+)}$ and $\partial_\nu^j \rho^{(+)} = \partial_\nu^j \tilde{\rho}^{(+)}$ on Γ for all $j = 0, 1, 2, \dots$*

Proof. By Lemma 5.7, we recover M_R over Γ_- based on our assumptions. We then apply [4, Theorem 1.5]. ■

Corollary 5.9. *Suppose that $\Sigma_q \subset \Gamma$ and $c_{P/S} = \tilde{c}_{P/S}, \rho = \tilde{\rho}$ outside $\bar{\Omega}_q$. Assume $\mathcal{F} = \tilde{\mathcal{F}}$. Then $M_T \equiv \tilde{M}_T$ and $T \equiv \tilde{T}$ at Γ .*

5.1.3. Recovery of subsurface travel times and lens relations. Next, we show that one can recover the subsurface lens relations when knowing the parameters outside the domain Ω_q and the outside measurement operator.

Lemma 5.10. *Let $(x, \xi) \in \partial T^*\Omega_q \cap S_+^*\Omega_q \cap \mathcal{S}$, and assume the extended convex foliation condition. If $\mathcal{F} = \tilde{\mathcal{F}}$ and $\lambda = \tilde{\lambda}, \mu = \tilde{\mu}, \rho = \tilde{\rho}$ outside Ω_q , then $c_{P/S}$ and $\tilde{c}_{P/S}$ have identical q -interior lens relations w.r.t. Σ_q in a neighborhood of (x, ξ) within $T_{\Sigma_q}^*\Omega$.*

Proof. The proof is identical to that of [6, Lemma 5.6]. In that paper, the trivial density assumption was only needed to have $P = \tilde{P}$ outside $\bar{\Omega}_q$, which is also true in our case due to the assumptions given. In fact, the lemma is true even without the density assumption by the proof of [28, Theorem 10.2]. ■

We can combine the above lemma with Theorem 5.1 to obtain the key corollary. First, let $d_{P/S}^q$ denote the P/S -distance function restricted to $\bar{\Omega}_q \times \bar{\Omega}_q$, so that $d_{P/S}^q|_{\Sigma_q \times \Sigma_q}$ is the boundary distance function of Ω_q .

Corollary 5.11. *With the assumptions in the above lemma, if $d_{P/S}^q|_{\Sigma_q \times \Sigma_q} = \tilde{d}_{P/S}^q|_{\Sigma_q \times \Sigma_q}$ in some neighborhood of $x \in \Sigma_q$, then $c_{P/S} = \tilde{c}_{P/S}$ in some neighborhood of x .*

5.1.4. Unique continuation for elliptic operators. Consider the differential inequality

$$(5.2) \quad |\Delta^n u| \leq f(x, u, Du, \dots, D^k u),$$

where f is Lipschitz, $k = [3n/2]$, and $D^j u$ is the matrix that contains all partial derivatives of u of order j

We have a unique continuation result from [19, section 3].

Theorem 5.12 (Protter theorem). *If u satisfies (5.2) in a neighborhood D of the origin and u vanishes in any neighborhood of the origin (not necessarily D), then u vanishes identically on D . In fact, the conclusion holds if*

$$e^{2/|x|^\alpha} u \rightarrow 0 \text{ as } |x| \rightarrow 0$$

for any positive α .

In the main proof, we will eventually recover “lower order” amplitudes of elastic waves and use them to show $\rho - \tilde{\rho}$ satisfies an elliptic, fourth order PDE and an inequality of the form (5.2). We can then use the Protter theorem to locally determine ρ .

Following the sketch of the proof at the start of section 5, we divide the proof into two parts. The first part concerns previous results on local travel time tomography that is used to recover the subsurface lens relation at each Σ_τ from the knowledge of the parameters in Ω_τ^* . We also analyze reflected amplitudes to recover the jumps in the material parameters and their derivatives across Σ_τ if there is an interface there. We then recover the amplitudes of P -waves along the bicharacteristic curves in a smooth neighbourhood of Σ_τ . The second part deals with the analysis of the amplitudes along the bicharacteristic curves in a smooth neighborhood of Σ_τ . Using the lower order terms along with the principal part of the symbol $p(t, x, \tau, \xi)$, we obtain a transport equation for the amplitude, which in turn helps us to recover the density.

5.2. Proof of Theorem 1.1 and absence of gauge freedom.

Proof of Theorem 1.1. The proof is by contradiction. Suppose $c_P \neq \tilde{c}_P$ or $c_S \neq \tilde{c}_S$ or $\rho \neq \tilde{\rho}$, and let $f = |c_P - \tilde{c}_P|^2 + |c_S - \tilde{c}_S|^2 + |\rho - \tilde{\rho}|^2$. Now consider $S := \Omega \cap \text{supp} f$, and take $q = \min_S \kappa$: so $c_P = \tilde{c}_P$ and $c_S = \tilde{c}_S$ and $\rho = \tilde{\rho}$ above Ω_q , but by compactness there is a point $x \in \Sigma_q \cap S$. The condition that $\kappa^{-1}(\tilde{q})$ has measure zero rules out the trivial case $q = \tilde{q}$.

First part. Let us now consider a small neighborhood of x , denoted B_x , and we consider the Σ_q -boundary distance function $d_{P/S}^q$ restricted to this neighborhood. Since the interfaces are not dense, we may choose B_x small enough to intersect at most one interface. We will consider P or S geodesics that stay completely inside B_x and connect boundary points of Ω_q . We will then construct microlocal P or S solutions in a small neighborhood of these geodesics as in [28, section 10].

The proof of [28, Theorem 10.2] ([6] has a slightly different proof) in conjunction with Corollary 5.2 or Theorem 5.1 shows $c_{P/S} = \tilde{c}_{P/S}$ in some neighborhood of x inside B_x which we keep denoting as B_x . Now let $(z_1, \zeta_1) \in \partial T^* \Omega_q$ with $z_1 \in B_x$ such that the ray $\gamma = \gamma_{P, z_1, \zeta_1}$

starting at (z_1, ζ_1) at time $t = 0$ remains inside B_x until it hits Ω_q again at some point $(z_2, \zeta_2) = L_{P,q}(z_1, \zeta_1)$ at time t_2 , recalling that $L_{P,q}$ is the q -interior lens relation for P -rays. Let Λ_t be the Lagrangian associated to the P -bicharacteristic flow of time t , but restricted to B_x . To describe the Lagrangian Λ_t , denote $\chi_t^P : T^*\mathbb{R}^n \rightarrow T^*\mathbb{R}^n$ the P -bicharacteristic flow by t units of time; that is, if $\alpha(t)$ is a smooth P -bicharacteristic with $\alpha(0) = (y, \eta)$, then $\chi_t^P(y, \eta) = \alpha(t)$. We can then write $\Lambda_t = \{(\chi_t^P(y, \eta), y, \eta); (y, \eta) \in T^*B_x\}$.

Let $V \in \mathcal{E}'(\Omega)$, the space of compactly supported distributions in Ω , with $\text{WF}(V) = \mathbb{R}_+(z_1, \zeta_1)$. As shown in [28, Proof of Theorem 10.2] using Proposition 5.4, we may construct an outgoing P wave u, \tilde{u} in Ω_q so that near z_1 , $u(T) \equiv V$ and $\tilde{u}(T) \equiv V$ at some time T . As in [28], γ does not hit an interface for either operator. We now consider two cases, depending on whether x is on an interface or not.

Smooth case: $x \notin \Gamma$. By the construction in [28] which uses Proposition 5.4, u and \tilde{u} are microlocal P -waves inside $B = [T, T + t_2] \times B_x$ and within this set are given by the forward propagator applied to V with wavefront set in Σ_P ,

$$(5.3) \quad u|_B \equiv \int e^{i\phi_P^+(t,x,\eta)} a_{+,k,P}^{:l}(t,x,\eta) \hat{V}_l(\eta) \, d\eta,$$

and likewise

$$(5.4) \quad \tilde{u}|_B \equiv \int e^{i\tilde{\phi}_P^+(t,x,\eta)} \tilde{a}_{+,k,P}^{:l}(t,x,\eta) \hat{V}_l(\eta) \, d\eta.$$

Since $\phi_P^+, \tilde{\phi}_P^+$ only depend on the wave speeds and we have recovered them inside B_x , these phase functions are equal within B_x . We would like to conclude that

$$(a_{+,k,P})_J - (\tilde{a}_{+,k,P})_J|_{\{(z_1, \zeta_1), (z_2, \zeta_2)\}} = 0$$

for each $J = 0, -1, -2, \dots$ in the polyhomogeneous expansion of the symbols.

Next, as shown in [6], $u \equiv \tilde{u}$ on Ω_q^c since $P = \tilde{P}$ outside Ω_q and using propagation of singularities. Thus,

$$\int e^{i\phi_P^+(t,x,\eta)} (a_{+,k,P}^{:l}(t,x,\eta) - \tilde{a}_{+,k,P}^{:l}(t,x,\eta)) \hat{V}_l(\eta) \, d\eta \in C^\infty$$

over Ω_q^c . Next, restrict these wave fields to $t = T + t_2 + \varepsilon$ and denote $t^* := t_2 + \varepsilon$ for some small $\varepsilon > 0$ that will go to 0 in the limit. Let us denote the symbol $b(x, \eta) = a_{+,k,P}(T + t^*, x, \eta) - \tilde{a}_{+,k,P}(T + t^*, x, \eta)$ restricted to B_x in the spatial variable. The leading order terms in the polyhomogeneous expansions of a and \tilde{a} only depend on the principal symbol of P, \tilde{P} and are thus equal since we have already recovered the wave speeds [28]. Thus, $b \sim \sum_{J=-1,-2,\dots} b_J$ where $b_{-1} = (a_{+,k,P}^{:l}(t^*, x, \xi))_{-1} - (\tilde{a}_{+,k,P}^{:l}(t^*, x, \xi))_{-1} \in S_{hom}^{-1}$, where S_{hom}^{-1} is the space of symbols of degree -1 that are homogeneous in the ξ coordinate (see also [15, Chapter 25]). Our goal is to conclude $b_{-1} = 0$ at (z_2, ζ_1) . Restricted to Ω_τ^c , we have

$$BV := \int e^{i\phi_P^+(T+t^*,x,\xi)} b(x, \xi) \hat{V}(\xi) \, d\xi = f \in C^\infty(B_x \cap \bar{\Omega}_q^c)$$

and B is an FIO associated to a canonical graph inside the class $\mathcal{I}^0(B_x \times B_x, \Lambda_{t^*})$. Note that $\Lambda_{t^*} = \{(\chi_{t^*}^P(y, \eta), y, \eta); (y, \eta) \in T^*B_x\}$, and in particular, $(\bar{z}, \bar{\zeta}) = \chi_{t^*}^P(z_1, \zeta_1)$. As shown in Appendix A, $\theta(y) := x \circ \chi_{t^*}^P(y, \zeta_1)$ is a local diffeomorphism from some neighborhood of z_1 to

a neighborhood of \bar{z} that we denote by W . We can shrink neighborhoods so that $W \subset \bar{\Omega}_\tau^c$. Let N denote the order of the distribution V (see Definition 4.1) and let $u \in \mathcal{C}_c^\infty(W)$. Denote $u_\tau(x) = \tau^{n/2} e^{-i\tau(x-\bar{z})\cdot\bar{\zeta}} u(\sqrt{\tau}(x-\bar{z}))$. Then Corollary 4.7 shows that

$$\liminf_{\tau \rightarrow \infty} \tau^{1-N} \langle BV, u_\tau \rangle = b_{-1}^*(\bar{z}, \zeta_2) J_{\theta^{-1}}(z_1) \liminf_{\tau \rightarrow \infty} \langle V \circ \theta^{-1}, u_\tau \rangle.$$

Also, $\lim_{\tau \rightarrow \infty} \langle f, u_\tau \rangle = 0$ since f is smooth by [31, Corollary 2.2.2]. We have the freedom to choose V so that this last limit is nonvanishing; we can even use the construction in [11, Example 2.6] so that the Fourier transform of V belongs in a certain 0th order symbol class so that $N = 0$. We conclude $b_{-1}(\bar{z}, \zeta_1) = 0$. Repeating this argument for $\varepsilon \rightarrow 0$ lets us conclude $b_{-1}(z_2, \zeta_1) = 0$ since $J_{\theta^{-1}}$ is nonvanishing. We may repeat this argument by adjusting the weight τ^{k-N} to show vanishing of the lower order terms of b as well but we do not need it.

This can be done for any such V and downward (z_1, ζ_1) near $T_x^* \partial \Omega_q$ so we conclude by (4.6)

$$a_{+,k,P}^l(t_2, z_2, \zeta_1) = \tilde{a}_{+,k,P}^l(t_2, z_2, \zeta_1) \text{ modulo } S^{-\infty}.$$

Interface case: $x \in \Gamma$. This case differs from the previous case since we want measurements of the amplitude on $\Sigma_+ \supset \Gamma_+$, while our assumptions only allow measurements on Γ_- . We are essentially treating Γ_+ (and not Γ_-) as the boundary for Ω_q . Fortunately, the difference is given by the transmission operator which we can recover via Corollary 5.9. Note that since we recovered the speeds, $\gamma = \tilde{\gamma}$.

As shown in [6], our inductive assumptions, propagation of singularities, and the foliation condition imply $u \equiv \tilde{u}$ in Ω_q^c since $P = \tilde{P}$ in that region. Near (T, z_1) , by construction we have $u|_{\Gamma_+} = M_T u|_{\Gamma_-}$ and $\tilde{u}|_{\Gamma_+} = \tilde{M}_T \tilde{u}|_{\Gamma_-}$. By Corollary 5.8 and since $u|_{\Gamma_-} \equiv \tilde{u}|_{\Gamma_-}$, we conclude $u|_{\Gamma_+} \equiv \tilde{u}|_{\Gamma_+} \equiv V$ near (T, z_1) .

By construction in Proposition 5.4, u and \tilde{u} are microlocal P -waves inside $B = [T, T+t_2] \times B_x$ as in the previous case and within this set are given by the forward propagator applied to V with wavefront set in Σ_P just as in (5.3) and (5.4) above. Let us provide a quick illustration of the argument we that are about to make. Consider [6, Figure 3], where essentially an elastic P wave is generated that is singular along the P -bicharacteristic through (x, ξ) and $L_{P,q}(x, \xi)$, and Ω_τ in the figure is Ω_q in our notation. This wave transmits from below Ω_τ to above Ω_τ in that picture as it passes $L_{P,q}(x, \xi)$. By induction, we have recovered this P -wave above Ω_τ , but we need to recover it just below Ω_τ since we are viewing Ω_τ as our unknown manifold in order to apply local tensor tomography results. The difference between the wave just below Ω_τ and above Ω_τ is determined by the transmission operator M_T , which we can recover using Corollary 5.9, which allows us to recover the P -wave just below Ω_τ .

Near $(T+t_2, z_2)$, we have again have $u|_{\Gamma_-} \equiv M_T u|_{\Gamma_+}$ and $\tilde{u}|_{\Gamma_-} \equiv \tilde{M}_T \tilde{u}|_{\Gamma_+}$. Since $u \equiv \tilde{u}$ outside $\bar{\Omega}_q$ and using Corollary 5.9, we conclude $u|_{\Gamma_+} \equiv \tilde{u}|_{\Gamma_+}$ near $(T+t_2, z_2)$ as well. Denote ρ_{Γ_+} as the restriction to Γ from below near z_2 . As shown in [12, Chapter 5], if $\text{WF}(u)$ over Γ_+ contains no covectors tangential to Γ , then $\rho_{\Gamma_+} u$ is also the image of an FIO in \mathcal{I}^0 (applied to V) associated to a canonical graph Lagrangian $\Lambda \subset T^*(\Gamma_+ \times \mathbb{R}_t) \times T^*B_x$. The Lagrangian can be described using χ_t^P from before. We compute $\Lambda = \{t(y, \eta), -|\eta|_P, d\rho_{\Gamma_+} \chi_{t(y, \eta)}^P(y, \eta), (y, \eta)\}$ where $t(y, \eta)$ is the time the P -ray from (y, η) hits Γ for the first time. It will be convenient to just define $\Phi(y, \eta) = (t(y, \eta), -|\eta|_P, d\rho_{\Gamma_+} \chi_{t(y, \eta)}^P)$. Let us introduce boundary normal

coordinates for Γ near z_2 with local coordinates (x', x_n) , where Γ is given by $x_n = 0$. Using (5.4) and (5.3), we then have near (t_2, z_2)

$$(5.5) \quad \rho_{\Gamma_+} u \equiv \int e^{i\phi_P^+(t, x', \eta)} a_{+,k,P}^{\cdot,l}(t, x', \eta) \hat{V}_l(\eta) d\eta$$

and likewise

$$(5.6) \quad \rho_{\Gamma_+} \tilde{u} \equiv \int e^{i\tilde{\phi}_P^+(t, x', \eta)} \tilde{a}_{+,k,P}^{\cdot,l}(t, x', \eta) \hat{V}_l(\eta) d\eta.$$

Analogous to the previous case, we define $b(t, x', \eta) = a_{+,k,P}(t, x', \eta) - \tilde{a}_{+,k,P}(t, x', \eta)$ restricted to a small neighborhood of (t_2, z_2) within $\mathbb{R}_t \times \Gamma_+$. As in the previous case, $b_0 = 0$ in the polyhomogeneous expansion of b . Restricted to $\bar{\Omega}_q^c$, we have as before

$$BV := \int e^{i\phi_P^+(t, x', \xi)} b(t, x', \xi) \hat{V}(\xi) d\xi = f \in C^\infty(\mathbb{R}_t \times \Gamma_+ \cap \mathbb{R}_t \times \bar{\Omega}_q^c),$$

where B is an FIO associated to the canonical graph Φ inside the class $\mathcal{I}^0(\mathbb{R}_t \times \Gamma_+ \times B_x, \Lambda)$. To apply the argument in the previous case with Proposition 4.6, all that is necessary is that B is an FIO associated to a canonical graph so that $(t, x') \circ \Phi(y, \eta)$ is a local diffeomorphism, since we are away from the glancing set. Hence, using the test functions $u_\tau(x) = \tau^{n/2} e^{-i\tau[(x' - z_2) \cdot \zeta_2 - (t - t_2)|\zeta_1|_P]} u(\sqrt{\tau}(t - t_2), \sqrt{\tau}(x' - z_2))$, we apply the argument in the previous case to conclude $b_{-1}(t_2, z_2, \zeta_1) = 0$. Likewise, we can iterate to show the vanishing of the lower order terms as well.

Hence, we are in the same situation as the smooth case above and the remaining argument proceeds as above to conclude $a_{\pm,k,P} = \tilde{a}_{\pm,k,P}$ on Λ (the principal amplitudes may be taken as functions on Λ via a diffeomorphism [15, Chapter 25]) modulo $S^{-\infty}$.

Second part. Above, we have concluded that $a_{\pm,k,P} = \tilde{a}_{\pm,k,P}$ modulo $S^{-\infty}$ when restricted to the P -bicharacteristic flow Lagrangian. We calculate the amplitude $a_{\pm,k,P}$ on the P bicharacteristic segment γ .

Recall the parametrization of the bicharacteristic curves as in (3.3). Let us evaluate the compatibility condition (3.12) for $J = 0, -1$. We use the notation that for two vectors, v and w , denote $v \otimes w := \frac{1}{2}(v \otimes w + w \otimes v)$. From a straightforward calculation we obtain (see [3, section 3])

$$(5.7) \quad \begin{aligned} \partial_{\tau, \xi} p(t, x, \tau, \xi) &= 2\rho \left(-\tau I, [c_S^2 \xi_1 I + (c_P^2 - c_S^2)(e_1 \otimes \xi)], \dots, [c_S^2 \xi_1 I + (c_P^2 - c_S^2)(e_3 \otimes \xi)] \right), \\ \partial_{\xi_j \xi_k} p(t, x, \tau, \xi) &= 2\rho \left[\delta_{jk} c_S^2 I + (c_P^2 - c_S^2)(e_3 \otimes e_k) \right], \\ N \cdot \partial_{t,x} N &= \frac{1}{2} \partial_{t,x} (N \cdot N) = 0, \end{aligned}$$

where N was defined right below (3.10).

Let us recall from (3.11) that $(a_{\pm,k,P}^{\cdot,l})_J = (h_{\pm,k,P}^{\cdot,l})_J + (b_{\pm,k,P}^{\cdot,l})_J N$ for $J = 0, -1, \dots$. Note that $(h_{\pm,k,P}^{\cdot,l})_0 = 0$ and let us denote $(b_{\pm,k,P}^{\cdot,l})_J = b_J$ for $J = 0, -1$. Using the fact that $(a_{\pm,k,P/S}^{\cdot,l})_1 = 0$, the compatibility condition (3.12) for $J = 0$ reduces to

$$\begin{aligned}
 0 &= N [B_P(b)_0] \\
 &= 2i\rho \left[-\tau\partial_t + c_P^2(\xi \cdot \nabla_x) + \frac{1}{2}(c_P^2 - c_S^2)|\xi|(\nabla_x \cdot N) + ic_P^2\xi \cdot \nabla_x(\log \rho c_P^2) \right] (b)_0 \\
 &\quad - i\rho \left[c_P^2 N(\nabla \otimes \xi)N + c_P^2 \frac{d}{ds}(\log c_P)|\xi| - c_S^2(\nabla_x \cdot \xi) - (c_P^2 - c_S^2)N(\nabla_x \otimes \xi)N \right] (b)_0.
 \end{aligned}$$

The above equation, combined with (3.3), reduces to a transport equation

$$\frac{d}{ds}(b)_0 = -\frac{1}{2} \left[\frac{d}{ds} \log(\rho c_P) + (\nabla_x \cdot N) \right] (b)_0.$$

For $s > T_0$, we solve the transport equation above and get (see [3, equation 16])

$$(5.8) \quad (b)_0(s) = (b)_0(0) \sqrt{\frac{(\rho c_P)|_{s=0}}{(\rho c_P)(s)}} \exp\left(-\frac{1}{2} \int_0^s (\nabla_x \cdot N)(r) dr\right) \quad \text{and} \quad (a)_0 = N \otimes (b)_0.$$

For $J = -1$, the compatibility condition (3.12) reads, recalling the definition of B_P and C_P in (3.10),

$$N[B_P(a)_{-1} + C_P(a)_0] = 0.$$

Using a similar calculation as for $J = 0$ we get

$$(5.9) \quad \frac{d}{ds}(a)_{-1} + \frac{1}{2} \left[\frac{d}{ds}(\log \rho c_P) + (\nabla_x \cdot N) \right] (a)_{-1} = G = \frac{1}{2i\rho c_P^2|\xi|} \left[NB_P(h_{\pm,k,P}^{i,l})_{-1} + NC_P(a)_0 \right].$$

Note that one can determine

$$(h_{\pm,k,P}^{i,l})_{-1} = \frac{-1}{\rho(c_P^2 - c_S^2)|\xi|^2} B_P(N \otimes (a)_0)$$

from (3.9). Thus, we can solve the above transport equation for $(a)_{-1}$ and get

$$(5.10) \quad g(a)_{-1} = C + \int_{\gamma} gG,$$

where C is a constant and

$$g(s) = \frac{\sqrt{(\rho c_P)(s)}}{(a)_0(T_0)\sqrt{(\rho c_P)(T_0)}} \exp\left(\int_0^s \nabla_x \cdot N(r) dr\right).$$

Now, note that U , M_T , and $\partial_\nu^j \lambda$, $\partial_\nu^j \mu$, $\partial_\nu^j \rho$ for $j = 0, 1, \dots$ are known at Σ_τ^\pm , thus we know $(a)_{-1}|_{s=T_0}$. Let there be two sets of parameters (λ, μ, ρ) and $(\tilde{\lambda}, \tilde{\mu}, \tilde{\rho})$ in Ω as assumed in the statement of Theorem 1.1. Let $(\tilde{a})_{-1}$ be the same quantity as $(a)_1$ corresponding to the parameters $(\tilde{\lambda}, \tilde{\mu}, \tilde{\rho})$. Therefore, from the above analysis we get $(a)_1(0) = (\tilde{a})_1(0)$. Since we

already have that the wave speeds $c_{P/S}$ are the same in a neighborhood of Σ_τ , the geodesics are also the same near Σ_τ . Since the projection of ζ in Ω is a geodesic γ , from (5.10) we obtain

$$(5.11) \quad \int_\gamma N \cdot \left(\frac{A - \tilde{A}}{c_P} \right) N ds = 0,$$

where A and \tilde{A} are exactly the same as the matrices A_1, A_2 derived in [3, equation 21]. Here A, \tilde{A} are 2-tensors and using the local inversion result [26] we obtain the kernel of the geodesic ray transform (5.11), given by the *Saint-Venant operator* (see [3]). The *Saint-Venant operator* applied on the 2-tensor $\frac{1}{c_P}(A - \tilde{A})$ results in a symmetric 4-tensor.

Using the exact same calculations of the symmetric 4-tensor as in [3, section 4] we finally obtain the fourth order PDE

$$(5.12) \quad \frac{(c_P^2 - c_S^2)(c_P^2 - 4c_S^2)}{c_P^4 - 5c_P^2c_S^2 + 8c_S^4} \Delta^2 \log \left(\frac{\rho}{\tilde{\rho}} \right) - \Delta \left(\nabla_x \log(\rho\tilde{\rho}) \cdot \nabla_x \log \left(\frac{\rho}{\tilde{\rho}} \right) \right) = 0$$

in a neighborhood of Σ_q in $\bar{\Omega}_q$. Since we know the derivatives of the parameters on Σ_q^+ by Corollary 5.8, we can smoothly extend them in a neighborhood of Σ_q in Ω_q^* . If we exclude the set $\mathcal{D} = c_P = 2c_S$, then (5.12) is elliptic and we can use strong unique continuation to prove $(\log \rho - \log \tilde{\rho}) = 0$ in the neighborhood of Σ_q .

In particular, we have shown

$$\beta^- := \log \rho - \log \tilde{\rho} = 0 \text{ in } B_x \cap \Omega_q.$$

By Corollary 5.8, β^- vanishes to infinite order at $\partial\Omega_q$, and so we may extend all three parameters smoothly outside Ω_q with β^- extended by 0. Let B_{ext} be a neighborhood of x such that $B_{ext} \cap \Omega_q = B$. With the extended parameters, β^- is extended as well, continues to satisfy (5.12), and in particular, over B_{ext} .

Now, if $F = \frac{(c_P^2 - c_S^2)(c_P^2 - 4c_S^2)}{c_P^4 - 5c_P^2c_S^2 + 8c_S^4}$ is bounded on B_{ext} , recalling that $D^j v$ denotes the j -tensor of partial derivatives of v of order j , then (5.12) shows that

$$|\Delta^2(\beta^-)| \leq Cg(x, D\beta^-, D^2\beta^-, D^3\beta^-)$$

so if we replace β^- by $\beta^-(x - x_0)$ for and x_0 in $B_{ext} \setminus \bar{\Omega}_q$, then β^- indeed satisfies an inequality of the form (5.2) as well as the hypothesis of Theorem 5.12. Hence, $\beta^- = 0$ on B_{ext} .

Thus, we can recover ρ in a neighborhood of Σ_q , excluding the set \mathcal{D} . Since we already have recovered $c_{P/S}$, we can recover all the parameters near Σ_q . Thus, $S = \emptyset$, which proves the theorem. ■

6. Conclusion. This paper implies that under certain geometric conditions, a piecewise smooth coefficient of a hyperbolic partial differential operator that is not in its principal symbol may be uniquely recovered. The essential ingredient is having a microlocal parametrix to represent solutions via an FIO in order to recover travel times and “lower order” polarization terms. This reduces the problem to a local tensor tomography problem. Even the scattering control construction may be generalized to whenever the transmission operator is an elliptic

operator, which depends on the original operator and the transmission conditions (see [6] for the construction).

However, this does not automatically lead to a unique recovery since the tensor tomography problem always has a gauge freedom. Thus, determining whether the gauge freedom may be eliminated will have to be done on a case by case basis that will be unique to the partial differential operator at hand.

Future work concerns both the reconstruction aspects of this inverse problem and the missing stability estimates discussed in Remark 1.4.

Appendix A. Proofs of statements in section 4. In this section, we provide the proofs of the statements made in section 4. First we prove Lemma 4.3 on the principal symbol of a PsiDO applied to a distribution.

Proof of Lemma 4.3. Let P^* be the formal distributional adjoint of P with symbol denoted p^* . Then

$$\begin{aligned} \langle (Pf)_\phi^\tau, u \rangle &= \langle Pf, e^{-i\tau x \cdot \xi} u(\sqrt{\tau}x) \rangle \tau^{n/2} \\ &= \langle f, P^* e^{-i\tau x \cdot \xi} u(\sqrt{\tau}x) \rangle \tau^{n/2}, \\ P^* e^{-i\tau x \cdot \xi} u(\sqrt{\tau}x) &= c_n \int e^{i(x-y) \cdot \eta - i\tau y \cdot \xi} p^*(x, \eta) u(y\sqrt{\tau}) dy d\eta \\ &= c_n \tau^{-n/2} \int e^{ix \cdot \eta - i \frac{y}{\sqrt{\tau}} \cdot (\eta + \tau\xi)} p^*(x, \eta) u(y) dy d\eta \\ &= c_n \tau^{-n/2} \int e^{ix \cdot \eta - iy \cdot (\eta/\sqrt{\tau} + \sqrt{\tau}\xi)} p^*(x, \eta) u(y) dy d\eta, \end{aligned}$$

where we changed variables $\sqrt{\tau}y = y$ in the last line. Change variables

$$\tilde{\eta} = \eta/\sqrt{\tau} + \sqrt{\tau}\xi$$

so that

$$\eta = \tilde{\eta}\sqrt{\tau} - \tau\xi$$

and $d\eta = \tau^{n/2} d\tilde{\eta}$. Substituting gives

$$\begin{aligned} P^* e^{-i\tau x \cdot \xi} u(\sqrt{\tau}x) &= c_n \int e^{ix \cdot (\tilde{\eta}\sqrt{\tau} - \tau\xi) - iy \cdot \tilde{\eta}} p^*(x, \sqrt{\tau}\tilde{\eta} - \tau\xi) u(y) dy d\tilde{\eta} \\ &= c_n e^{-i\tau x \cdot \xi} \int e^{i(x\sqrt{\tau} - y) \cdot \tilde{\eta}} p^*(x, \sqrt{\tau}\tilde{\eta} - \tau\xi) u(y) dy d\tilde{\eta} \\ &:= e^{-i\tau x \cdot \xi} v(\sqrt{\tau}x), \end{aligned}$$

where $v^\tau(x) = c_n \int e^{i(x\sqrt{\tau} - y) \cdot \tilde{\eta}} p^*(\frac{x}{\sqrt{\tau}}, \sqrt{\tau}\tilde{\eta} - \tau\xi) u(y) dy d\tilde{\eta}$ which is a PsiDO applied to u . Hence, we have shown that

$$\langle (Pf)_\phi^\tau, u \rangle = \langle (f)_\phi^\tau, v^\tau(x) \rangle,$$

i.e., the family $(f)_\phi^\tau$ applied to the “test function” $v^\tau(x)$ which also depends on τ .

Now suppose P is a classical PsiDO so p^* is homogeneous of degree m . Let us do the first order Taylor expansion of $p^*\left(\frac{x}{\sqrt{\tau}}, \sqrt{\tau}\tilde{\eta} - \tau\xi\right)$ around $x = 0$ and $\tilde{\eta} = 0$:

$$(A.1) \quad p^*\left(\frac{x}{\sqrt{\tau}}, \sqrt{\tau}\tilde{\eta} - \tau\xi\right) = p^*(0, -\tau\xi) + \tau^{-1/2}x \cdot \partial_x p^*(0, -\tau\xi) + \tau^{1/2}\tilde{\eta} \cdot \partial_\eta p^*(0, -\tau\xi).$$

However, note that $\partial_\eta p^* \in \partial_\eta p_m^* + S_{hom}^{m-2} = \tau^{m-1}\partial_\eta p_m^*(0, -\xi) + S_{hom}^{m-2}$. Thus,

$$p^*\left(\frac{x}{\sqrt{\tau}}, \sqrt{\tau}\tilde{\eta} - \tau\xi\right) = p^*(0, -\tau\xi) + O(\tau^{m-1/2}),$$

so we obtain

$$v^\tau(x) = p^*(0, -\tau\xi)u(x) + O(\tau^{m-1/2}).$$

We have now shown

$$\langle (Pf)_\phi^\tau, u \rangle = \langle p^*(0, -\tau\xi)(f)_\phi^\tau, u(x) \rangle + O(\tau^{N+m-1/2}),$$

which gives us the desired result on the principal symbol.

For the latter statement in the lemma, we have

$$\begin{aligned} P e^{-i\tau(x-x_0)\cdot\xi} u(\sqrt{\tau}(y-x_0)) &= c_n \int e^{i(x-y)\cdot\eta - i\tau(y-x_0)\cdot\xi} p(x, \eta) u(y\sqrt{\tau}) dy d\eta \\ &= c_n \tau^{-n/2} \int e^{i((x-x_0)-y/\sqrt{\tau})\cdot\eta - i\sqrt{\tau}y\cdot\xi} p(x, \eta) u(y) dy d\eta \\ &= c_n \tau^{-n/2} \int e^{i\tilde{x}\cdot\eta - i\frac{y}{\sqrt{\tau}}\cdot(\eta+\tau\xi)} p(x, \eta) u(y) dy d\eta \\ &= c_n \tau^{-n/2} \int e^{i\tilde{x}\cdot\eta - iy\cdot(\eta/\sqrt{\tau} + \sqrt{\tau}\xi)} p(x, \eta) u(y) dy d\eta, \end{aligned}$$

where we changed variables $\sqrt{\tau}(y-x_0) = y$ in the second line and we denote $\tilde{x} = x - x_0$. Change variables

$$\tilde{\eta} = \eta/\sqrt{\tau} + \sqrt{\tau}\xi$$

so

$$\eta = \tilde{\eta}\sqrt{\tau} - \tau\xi,$$

and $d\eta = \tau^{n/2}d\tilde{\eta}$. Substituting gives

$$\begin{aligned} P e^{-i\tau(x-x_0)\cdot\xi} u(\sqrt{\tau}(y-x_0)) &= c_n \int e^{i\tilde{x}\cdot(\tilde{\eta}\sqrt{\tau} - \tau\xi) - iy\cdot\tilde{\eta}} p(x, \sqrt{\tau}\tilde{\eta} - \tau\xi) u(y) dy d\tilde{\eta} \\ &= c_n e^{-i\tau\tilde{x}\cdot\xi} \int e^{i(\tilde{x}\sqrt{\tau}-y)\cdot\tilde{\eta}} p(x, \sqrt{\tau}\tilde{\eta} - \tau\xi) u(y) dy d\tilde{\eta} \\ &:= e^{-i\tau\tilde{x}\cdot\xi} v(\sqrt{\tau}\tilde{x}), \end{aligned}$$

where $v^\tau(x) = c_n \int e^{i(x-y)\cdot\tilde{\eta}} p\left(\frac{x}{\sqrt{\tau}} + x_0, \sqrt{\tau}\tilde{\eta} - \tau\xi\right) u(y) dy d\tilde{\eta}$ and can be viewed as a parameter dependent PsiDO applied to u .

Now suppose P is a classical PsiDO so p is homogeneous of degree m . Let us do the first order Taylor expansion of $p(\frac{x}{\sqrt{\tau}} + x_0, \sqrt{\tau}\tilde{\eta} - \tau\xi)$ around $x = 0$ and $\tilde{\eta} = 0$.

$$(A.2) \quad p\left(\frac{x}{\sqrt{\tau}} + x_0, \sqrt{\tau}\tilde{\eta} - \tau\xi\right) = p(x_0, -\tau\xi) + \tau^{-1/2}x \cdot \partial_x p(x_0, -\tau\xi) + \tau^{1/2}\tilde{\eta} \cdot \partial_{\eta} p(x_0, -\tau\xi) + O(\tau^{-1}).$$

However, note that $\partial_{\eta} p = \partial_{\eta} p_m \bmod S^{m-2} = \tau^{m-1} \partial_{\eta} p_m(0, -\xi) \bmod S^{m-2}$. Since u is compactly supported, we also have

$$(A.3) \quad x \cdot \partial_x p(x_0, -\tau\xi) u(\sqrt{\tau}x) = O(\tau^{m-1/2})$$

since $|x| \lesssim \tau^{-1/2}$ on the support of u . The rigorous details follow the proof in [31, Theorem 2.1.2] with appropriate cutoff functions.

Thus,

$$p\left(\frac{x}{\sqrt{\tau}} + x_0, \sqrt{\tau}\tilde{\eta} - \tau\xi\right) = p(x_0, -\tau\xi) + O(\tau^{m-1/2}),$$

so we obtain

$$v^{\tau}(x) = p(x_0, -\tau\xi) u(x) + O(\tau^{m-1/2}).$$

Replacing x by $\sqrt{\tau}\tilde{x}$ and using (A.3) gives us exactly $Pu_{\tau} = p(x_0, -\tau\xi) u_{\tau} + O(\tau^{m-1/2})$. ■

We now prove Proposition 4.4 on the symbol of a pullback of a distribution by a diffeomorphism.

Proof of Proposition 4.4. This follows easily from the proof of [31, Proposition 1.4.1] once we make several observations. First, we have by construction

$$\begin{aligned} & \tau^{-n/2} \left(\langle (f \circ \theta)_{\phi \circ \theta}^{\tau}, u \rangle - \langle f_{\phi}^{\tau} \circ d_{x_0} \theta, u \rangle \right) \\ &= \tau^{-n/2} \left(\langle f_{\phi}^{\tau}, J_{\theta^{-1}}(y) u(\sqrt{\tau}(\theta^{-1}(y) - x_0)) \rangle - \langle f, e^{-i\tau\phi(y)} J_{\theta^{-1}}(y_0) u(\sqrt{\tau}(d_{x_0}\theta)^{-1}(y - y_0)) \rangle \right) \\ &= \tau^{-n/2} \langle f, e^{-i\tau\phi(y)} \left[J_{\theta^{-1}}(y) u(\sqrt{\tau}(\theta^{-1}(y) - x_0)) - J_{\theta^{-1}}(y_0) u(\sqrt{\tau}(d_{x_0}\theta)^{-1}(y - y_0)) \right] \rangle \\ &= \tau^{-n/2} \langle f, e^{-i\tau\phi(y)} v^{\tau}(\sqrt{\tau}(y - y_0)) \rangle, \end{aligned}$$

where

$$v^{\tau}(y) = J_{\theta^{-1}}(y/\sqrt{\tau} + y_0) u(\sqrt{\tau}(\theta^{-1}(y/\sqrt{\tau} + y_0) - x_0)) - J_{\theta^{-1}}(y_0) u((d_{x_0}\theta)^{-1}y).$$

Denote

$$A := d_{y_0}\theta^{-1} = (d_{x_0}\theta)^{-1}.$$

Introducing the variable $Y \in \mathbb{R}^n$, and using Taylor series, we have

$$[\theta^{-1}(Y + y_0)]_j = [\theta^{-1}(y_0)]_j + [AY]_j + \sum_{k,l=1}^n b_{j,kl}(Y) Y_k Y_l,$$

where $b_{j,..}(Y)$ is a smooth $n \times n$ matrix function, for $j = 1, \dots, n$.

Thus,

$$(A.4) \quad [\sqrt{\tau}(\theta^{-1}(y/\sqrt{\tau}) - x_0)]_k = [Ay]_j + \tau^{-1/2} \sum_{k,l=1}^n b_{j,kl}(y/\sqrt{\tau}) y_k y_l.$$

A Taylor expansion for u is

$$u(z+h) = u(z) + a(z, h) \cdot h,$$

where $a(z, h)$ is a smooth n -vector function of z and h , and similarly

$$J_{\theta^{-1}(y)}(y/\sqrt{\tau} + y_0) = |A| + \tau^{-1/2} C(y/\sqrt{\tau}) \cdot y,$$

where $C(Y)$ is a smooth n -vector function. Combining these equations with (A.4) we obtain

$$(A.5) \quad \begin{aligned} v^\tau(y) &= J_{\theta^{-1}(y)}(y/\sqrt{\tau} + y_0) u(Ay + \tau^{-1/2} \sum_{j,k,l=1}^n a_j(y, b_{j,kl}(y/\sqrt{\tau}) y_k y_l)) - |A| u(Ay) \\ &= \tau^{-1/2} \left[\sum_{j,k,l=1}^n a \left(Ay, \sqrt{\tau}(\theta^{-1}(y/\sqrt{\tau}) - x_0) \right) b_{j,kl} \left(\frac{y}{\tau} \right) y_k y_l \right] J_{\theta^{-1}(y)}(y/\sqrt{\tau} + y_0) \\ &\quad + \tau^{-1/2} u(Ay) C(y/\sqrt{\tau}) \cdot y. \end{aligned}$$

Hence, the above equation is analogous to [31, equation (1.4.9)] and the rest of the proof follows closely to [31, Proof of Proposition 1.4.1]. \blacksquare

We now prove Proposition 4.6 regarding the symbol of an FIO applied to a distribution.

Proof of Proposition 4.6. It will ease notation to use \tilde{A} in place of A in the statement of the proposition. We start with a test function $u \in \mathcal{D}(X)$ so

$$(A.6) \quad \langle (\tilde{A}f)_{x_0, \xi_0}^\tau, u \rangle = \langle \tilde{A}f, e^{-i\tau(x-x_0) \cdot \xi_0} u(\sqrt{\tau}(x-x_0)) \rangle \tau^{n/2}$$

$$(A.7) \quad = \langle f, \tilde{A}^t e^{-i\tau(x-x_0) \cdot \xi_0} u(\sqrt{\tau}(x-x_0)) \rangle \tau^{n/2}.$$

Set $A = \tilde{A}$ which is also an FIO associated to a canonical graph. An FIO associated to a canonical graph has the form

$$Au(y) = \int e^{iS(y, \xi)} a(y, \xi) \hat{u}(\xi) d\xi.$$

Denote $u_\tau = e^{i(x-x_0) \cdot \xi_0} u(\sqrt{\tau}(x-x_0))$, and

Now, we view u as a wave packet centered at x_0 around ξ_0 . We do the Taylor expansion

$$S(y, \xi) = S(y, \xi_0) + S'_\xi(y, \xi_0) \cdot (\xi - \xi_0) + q(y, \xi - \xi_0),$$

where q is quadratic in $\xi - \xi_0$. By homogeneity, $S(y, \xi_0) - S'_\xi(y, \xi_0) \cdot \xi_0 = 0$. So

$$S(y, \xi) = S'_\xi(y, \xi_0) \cdot \xi + q(y, \xi - \xi_0).$$

Thus,

$$Au(y) = \int e^{iS'_\xi(y, \xi_0) \cdot \xi} e^{iq(y, \xi)} a(y, \xi) \hat{u}(\xi) d\xi.$$

Next, χ being a canonical graph implies that the map

$$y \mapsto S'_\xi(y, \xi_0) = x(y, \xi_0) := T(y)$$

is a diffeomorphism near $y_0 = \pi_1 \chi(x_0, \xi_0)$, where π_1 is the projection to the base manifold. Applying a pullback via the inverse of the above map gives

$$Au(y(x)) = \int e^{ix \cdot \xi} e^{iq(y(x), \xi)} a(y(x), \xi) \hat{u}(\xi) d\xi = B(x, D)u,$$

where B is a PsiDO with symbol $b(x, \xi) = e^{iq(y(x), \xi)} a(y(x), \xi)$. Technically, b is not a symbol but it will be after introducing cutoff functions as in the proof of [31, Theorem 3.2.5]. Denote $u_\tau = e^{i(x-x_0) \cdot \xi_0} u(\sqrt{\tau}(x-x_0))$ and

$$Au_\tau(y(x)) = \int e^{ix \cdot \xi} e^{iq(y(x), \xi)} a(y(x), \xi) \hat{u}_\tau(\tau\xi) d\xi.$$

As in the proof of [31, Theorem 3.2.5], we divide the above integral into two pieces with a cutoff function $\mathcal{K}(\Psi)$, $\Psi = \xi - \xi_0$ which is 1 for $|\Psi| \leq 1/2$ and supported in the unit ball. Thus, without loss of generality, we can replace b with $\mathcal{K}(\tau^{1/2}\Psi)b$, with the piece containing $1 - \mathcal{K}(\tau^{1/2}\Psi)$ being $O(\tau^{-\infty})$ via the proof of [31, Theorem 3.2.5] and it can be ignored. Thus,

$$Au_\tau(y) = B(x, \tau D)u_\tau|_{x=x(y)},$$

where $b(x, \xi) = \mathcal{K}(\tau^{1/2}\Psi)e^{iq(y(x), \xi - \xi_0)} a(y(x), \xi)$ is now a symbol and $B(x, D)$ is a (non-classical) PsiDO with principal symbol of order μ denoted $b_\mu(y(x), \xi)$ (see, for example, [10]). We also denote $a_\mu(y, \xi)$ as the first term in the asymptotic expansion of $a(y, \xi)$, which is a classical symbol. This shows that Au_τ is merely an application of a PsiDO followed by a diffeomorphism. Due to the cutoff \mathcal{K} , the proof of Lemma 4.3 goes through. We showed that to leading order

$$B(x, D)u_\tau = b_\mu(x_0, \tau\xi_0)u_\tau(x) + O(\tau^{-1/2}).$$

Thus, we get $Au_\tau = e^{iq(y_0, \xi_0)} a_\mu(y_0, \xi_0)u_\tau(x(y)) + O(\tau^{-1/2})$.

Next observe that

$$u_\tau(x(y)) = e^{i\tau(x(y)-x_0) \cdot \xi_0} u(\sqrt{\tau}(x(y)-x_0))\tau^{n/2}$$

and note that the phase $\phi(y) = (x(y) - x_0) \cdot \xi_0 = (S'_\xi(y, \xi_0) - x_0) \cdot \xi_0 = S(y, \xi_0) - x_0 \cdot \xi_0$ satisfies the conditions in [31] for the phase, so that the phase invariance results there apply. We have $\phi(y_0) = 0$ and $\phi'(y_0) = S'_y(y_0, \xi_0) = \eta_0$, so [31, Proposition 1.2.1] shows we may replace ϕ by the phase $(y - y_0) \cdot \eta_0$ without changing the principal symbol.

Let θ be defined such that $\theta^{-1}(y) = T(y)$. Thus, using Proposition 4.4 we have

$$(A.8) \quad \langle g, u_\tau(x(y)) \rangle = \langle J_{\theta^{-1}}^{-1} g, J_{\theta^{-1}} u_\tau(x(y)) \rangle$$

$$(A.9) \quad = \langle (J_\theta g \circ \theta)_{\phi \circ \theta}^\tau, u(x) \rangle = \langle (J_\theta g)_{\phi}^\tau \circ d_{x_0} \theta, u(x) \rangle + O(\tau^{N-1/2}).$$

We also compute

$$(A.10) \quad \sigma_{\chi(y_0, \eta_0)}(J_\theta g \circ \theta) = \sigma_{y_0, \eta_0}(\theta)[\sigma_{y_0, \eta_0}(J_\theta g)] = J_\theta(y_0) \sigma_{y_0, \eta_0}(\theta)[\sigma_{y_0, \eta_0}(g)]$$

using that multiplication by J_θ can be viewed as an application of a 0th order classical PsiDO. Piecing everything together gives us

$$\sigma_{x_0, \xi_0}(\tilde{A}g) = a(y_0, \tau \xi_0) J_{T^{-1}}(y_0) \sigma_{y_0, \eta_0}(T^{-1})[\sigma_{y_0, \eta_0}(g)]. \quad \blacksquare$$

Appendix B. Elastic wave parametrix with scattering. In this section, we summarize the microlocal parametrix construction for the system (2.1) with transmission conditions that was used to prove Proposition 5.4 to recover the wave speeds, which was already done in [6].

B.1. Cauchy data and propagator. Recall the space \mathbf{C} in (2.4), where we define the Cauchy data. Let $\Psi = (\psi_0, \psi_1) \in \mathbf{C}$ be some Cauchy data. Observe that for $\Psi \in \mathbf{C}$, there exists a unique solution $U = U_\Psi \in C(\mathbb{R}, H^1(\mathbb{R}^3))$ of the initial value problem (2.1). We define the *propagator*

$$F : \mathbf{C} \rightarrow C(\mathbb{R}, H^1(\mathbb{R}^3)) \quad \text{as} \quad F(\Psi) := U(t, x) \quad \text{in} \quad \mathbb{R} \times \mathbb{R}^3.$$

The Cauchy propagation operator is defined as

$$(B.1) \quad \mathcal{F}_t : \mathbf{C} \mapsto \mathbf{C} \quad \text{as} \quad \mathcal{F}_t \Psi := r_t \circ F(\Psi) = (U(t, \cdot), \partial_t U(t, \cdot)),$$

where $U = U_\Psi$ and r_t is the restriction at time t .

Let us define λ_j, μ_j, ρ_j to be the smooth extensions of the parameters $(\lambda, \mu, \rho)|_{\Omega_j}$ outside Ω_j so that the solution operators E_0 and E_1 may be defined for each set of such parameters. We define the *Cauchy-to-solution* operators $\mathbf{J}_{\mathbf{C} \rightarrow \mathbf{S}}$ and $\mathbf{J}_{\mathbf{C} \rightarrow \mathbf{S}_+}$ as in [6, section 4.1]. Loosely speaking, $\mathbf{J}_{\mathbf{C} \rightarrow \mathbf{S}}$ maps the Cauchy data from \mathbf{C} to the unique solution $U_\Psi \in C(\mathbb{R}, H^1(\Omega))$ and $\mathbf{J}_{\mathbf{C} \rightarrow \mathbf{S}_+} U$ the same as $\mathbf{J}_{\mathbf{C} \rightarrow \mathbf{S}} U$ but only propagates forward in time. We also borrow the following operators from [6]:

$\mathbf{J}_{\mathbf{C} \rightarrow \partial}$: maps Cauchy data from \mathbf{C} to the boundary $C(\mathbb{R}, H^{1/2}(\partial\Omega_j))$,

$\mathbf{J}_{\mathbf{C} \rightarrow \partial_+}$: maps Cauchy data from \mathbf{C} to the boundary $C(\mathbb{R}, H^{1/2}(\partial\Omega_j))$,

$\mathbf{J}_{\partial \rightarrow \mathbf{S}}$: maps boundary data $C(\mathbb{R}, H^{1/2}(\partial\Omega_{j+1}))$ to the solution $C(\mathbb{R}, H^1(\Omega))$,

$\mathbf{J}_{\partial \rightarrow \partial}$: maps boundary data $C(\mathbb{R}, H^{1/2}(\partial\Omega_j))$ to a different boundary $C(\mathbb{R}, H^{1/2}(\partial\Omega_{j+1}))$.

B.2. P/S polarization projections. Let us construct a P/S-mode projector $\Pi_{P/S}$ that microlocally projects the elastic wave field u to the compressive (P) and the shear (S) wave fields for a small time interval, as

$$\Pi_{P/S} u := \sum_{\pm, l, k} \int_{\mathbb{R}^3} e^{i\phi_{P/S}^\pm(t, x, \xi)} a_{\pm, k, P/S}^{\cdot, l}(t, x, \xi) (\hat{\phi}_k)_l(\xi) d\xi, \quad \text{where } l = 1, 2, 3, \quad k = 1, 2.$$

Observe that $p(t, x, \tau, \xi)$ has eigenvalues $\rho(\tau^2 - c_P^2|\xi|^2)$ and $\rho(\tau^2 - c_S^2|\xi|^2)$ with multiplicity 1 and 2, respectively (see (3.1)). The matrix p can be diagonalized and there exists a unitary matrix $V(t, x, \tau, \xi)$ such that

$$Vp(t, x, \tau, \xi)V^{-1} = \rho \begin{pmatrix} \tau^2 - c_P^2|\xi|^2 & 0 & 0 \\ 0 & \tau^2 - c_S^2|\xi|^2 & 0 \\ 0 & 0 & \tau^2 - c_S^2|\xi|^2 \end{pmatrix} = D(t, x, \tau, \xi).$$

Let us now consider the symbol

$$(B.2) \quad \Pi_P(t, x, \tau, \xi) := V^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} V \quad \text{and} \quad \Pi_S(t, x, \tau, \xi) := V^{-1} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} V.$$

Observe that the symbol of $\Pi_{P/S}$ is homogeneous of order 0 in $|\xi|$ and thus $\Pi_{P/S}$ represents a 0th order PsiDO.

B.3. Transmission conditions. Let u_I be an incoming wave starting at $\Omega_{t_j}^*$ and traveling toward $\Gamma_j = \Sigma_{t_j}$ for $j = 0, 1, \dots, m$. At Γ_j it hits the interface and breaks into two parts u_R the reflected wave and u_T the transmitted wave. From now on, we will write the subscript $\bullet = I/R/T$ to denote the incoming, reflected, or transmitted quantities.

Let us define the Neumann operator at an interface Γ_j for $j = 0, 1, \dots, m$ as

$$(B.3) \quad \mathcal{N}_\bullet u_\bullet = (\lambda \operatorname{div} \otimes \mathbf{I} + 2\mu \widehat{\nabla}) u_\bullet \cdot \nu_\bullet|_\Gamma,$$

where ν is an outward unit normal vector at Γ_j . The elastic transmission conditions on the interface Γ_j for $j = 0, 1, \dots, m$ are given as

$$(B.4) \quad \begin{aligned} u_I + u_R &= u_T, \\ \mathcal{N}_I u_I + \mathcal{N}_R u_R &= \mathcal{N}_T u_T. \end{aligned}$$

The system (B.4) can be microlocally inverted to obtain the reflection and the transmission operators M_R and M_T , where $M_R u_I|_{\Gamma_j} = u_R$ and $M_T u_I|_{\Gamma_j} = u_T$. Note that the operators M_R, M_T are PsiDOs of order 0 on $\mathbb{R} \times \Gamma_j$ and have been calculated explicitly in [6]. The operators M_R, M_T change from interface to interface, but for the sake of notational simplicity we do not mention the influence of $j = 0, 1, \dots, m$ on them.

B.4. Parametrix. Define the operator $\iota : \Sigma_{t,\pm} \rightarrow \Sigma_{t,\pm}$ changes from one boundary to its copy in $\Sigma_{t,\pm}$. Consider the boundary operator $M = M_R + \iota M_T$. To understand the propagation of the wave field through this broken medium let us consider

$$(B.5) \quad \tilde{F} := \mathbf{J}_{\mathbf{C} \rightarrow \mathbf{S}} + \mathbf{J}_{\partial \rightarrow \mathbf{S}} \sum_{k=0}^{\infty} (\mathbf{J}_{\partial \rightarrow \partial} M)^k \mathbf{J}_{\mathbf{C} \rightarrow \mathbf{S}},$$

$$(B.6) \quad \tilde{R}_{2T} = r_{2T} \circ \tilde{F},$$

where r_{2T} is a restriction to $t = 2T$. Again omitting the proof, it can be shown that $\tilde{F} \equiv F$ and $\tilde{R}_{2T} \equiv R_{2T}$ away from glancing rays. In the elastic case it means that $\tilde{R}_{2T} h_0 \equiv R_{2T} h_0$ for

initial Cauchy data h_0 such that every broken bicharacteristic originating in $\text{WF}(h_0)$ is disjoint from both the P and S glancing sets described in [28]. Recalling that $M = M_R + M_T$, we may write \tilde{R}_{2T} as a sum of graph FIO indexed by sequences of reflections and transmissions:

$$(B.7) \quad \tilde{R}_{2T} = \sum_{s \in \{R, T\}^k, \lambda \in \{P, S\}^{k+1}} \tilde{R}_{s, \lambda}, \quad \tilde{R}_() = r_{2T} J_{\mathbf{C} \rightarrow \mathbf{S}},$$

$$(B.8) \quad \tilde{R}_{(s_1, \dots, s_k; \lambda_0, \dots, \lambda_k)} = r_{2T} J_{\partial \rightarrow \mathbf{S}} \Pi_{\lambda_k} M_{s_k} J_{\partial \rightarrow \partial} \cdots \Pi_{\lambda_2} M_{s_2} J_{\partial \rightarrow \partial} \Pi_{\lambda_1} M_{s_1} J_{\mathbf{C} \rightarrow \partial} \Pi_{\lambda_0}.$$

The solution operator \tilde{F} likewise decomposes into analogous components \tilde{F}_a .

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